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ENERGY DECAY AND SELF-PRESERVING CORRELATION FUNCTIONS IN ISOTROPIC TURBULENCE*

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1. Introduction. We consider those properties of turbulence in fluid of zero mean motion which can be deduced from the assumptions of spatial homogeneity and isotropy. The i-components of the velocity fluctuations at two points $P(\mathbf{x})$ and $P'(\mathbf{x}')$ will be written as u_i and u_i'. The spatial separation of P and P' is denoted by the vector

$$\xi = \mathbf{x}' - \mathbf{x}$$

of magnitude r, where

$$\xi_i \xi_i = r^2$$

and repeated indices imply summation over the values 1, 2 and 3.

It may be shown without difficulty[1,2] that the condition of isotropy requires the correlation between u_i and u_j', where i and j have arbitrary values, to depend only on the geometrical configuration defined by ξ and unit vectors in the i- and j-directions, and a single scalar function of r^2. It can be represented as the typical component of a tensor of the second rank in the following manner:

$$\overline{u_i u_j'} = R_i^j = u'^2\left[-\frac{1}{2r}\frac{\partial f}{\partial r}\xi_i\xi_j + \left(f + \frac{1}{2}r\frac{\partial f}{\partial r}\right)\delta_{ij}\right] \tag{1.1}$$

where the over-bar denotes a spatial mean value, $f(r)$ is the scalar function, and

$$u'^2 = \overline{u_1^2} = \overline{u_2^2} = \overline{u_3^2};$$

δ_{ij} is the unit tensor whose value is 1 if $i = j$, and 0 otherwise. The triple correlation between velocity components at P and P' has a similar dependence on a single scalar function;

$$\overline{u_i u_j u_k'} = T_{ij}^k$$

$$= u'^3\left[\left(\frac{k - r\,\partial k/\partial r}{2r^3}\right)\xi_i\xi_j\xi_k + \left(\frac{k + \frac{1}{2}r\,\partial k/\partial r}{2r}\right)(\delta_{ik}\xi_j + \delta_{kj}\xi_i) - \frac{k}{2r}\delta_{ij}\xi_k\right]. \tag{1.2}$$

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[1] T. v. Kármán and L. Howarth, *On the statistical theory of isotropic turbulence*, Proc. Roy. Soc. (A) **164**, 192-215 (1938).

[2] H. P. Robertson, *The invariant theory of isotropic turbulence*, Proc. Camb. Phil. Soc. **36**, 209-223 (1940).

The scalar function $k(r)$ is in this case an odd function of r, of order r^3 when r is small. It is clear from (1.1) and (1.2) that $f(r)$ and $k(r)$ are the correlation coefficients for particular values of the indices and the ξ_i, viz.

$$u'^2 \cdot f(r) = \overline{(u_i u_j')}_{\substack{i=j \\ \xi_i = r}},$$

$$u'^3 \cdot k(r) = \overline{(u_i u_j u_k')}_{\substack{i=j=k \\ \xi_i = r}}.$$

All higher order correlations which involve at least one velocity component at each of the points P and P' depend on more than one scalar function of r.

The equations of motion of a viscous incompressible fluid contain both linear and quadratic terms in the velocity components, and it is consequently possible to relate the double and triple velocity correlations. Expressed in terms of the scalar functions $f(r)$ and $k(r)$, this relation becomes the equation for the propagation of the double velocity correlation and has the form

$$\frac{\partial(u'^2 f)}{\partial t} = u'^3\left(k' + \frac{4}{r}k\right) + 2\nu u'^2\left(f'' + \frac{4}{r}f'\right), \tag{1.3}$$

where dashes to f and k denote differentiation with respect to r, and t is the time. Equation (1.3) has a simple and useful form for the particular value $r = 0$, when it describes the rate of decay of energy of the turbulence;

$$\frac{du'^2}{dt} = 10\nu u'^2(f'')_{r=0} = -\frac{10\nu u'^2}{\lambda^2} \tag{1.4}$$

where λ is the length parameter previously introduced by Taylor.[3]

Since the difficulties of measurement become very great as the order of the correlation increases, it is inevitable that Eq. (1.3) should occupy an important place in any practical theories of turbulence. The purpose of this paper is to extract as much information from it as is possible with a minimum of further assumptions, and in particular to deduce the rates of energy decay which are consistent with certain types of behaviour of the function $f(r)$. It is important to appreciate that Eq. (1.3) represents all the information about the function $f(r)$ which we possess, and is clearly insufficient to permit a complete solution to be obtained. This lack of information for the determination of correlation functions is inherent in averaged equations, and is the penalty paid for this use of the statistical method. Our plan is therefore (a) to discuss suitable limiting cases for which a solution of (1.3) is possible, and (b) to discuss the consequences of simple hypotheses so that their validity can be put to the test of experiment. Types of solution for which the function f is "self-preserving", i.e. for which f is a function only of r/L, where L is a length which depends on the time t, were introduced by v. Kármán and Howarth.[1] The consequent gain in simplicity of the mathematics is considerable and it is with such solutions that we shall largely be concerned here. There is available sufficient experimental evidence to indicate that the theoretical solutions correspond in some respects with reality and are worth pursuing.

2. Loitsiansky's invariant. We shall have need later of a simple deduction from the

[3]G. I. Taylor, *Statistical theory of turbulence*, Proc. Roy. Soc. (A) **151**, 421-478 (1935).

basic Eq. (1.3), which was pointed out first by Loitsiansky.[4] Multiplying both sides of the equation by r^4 and integrating over the range 0 to ∞, we find

$$\frac{\partial}{\partial t}\left(u'^2 \cdot \int_0^\infty r^4 f \, dr\right) = 0$$

provided that $r^4 f' \to 0$ and $r^4 k \to 0$ as $r \to \infty$. Physically it seems reasonable to suppose that the convergence conditions are satisfied; then

$$u'^2 \cdot \int_0^\infty r^4 f \, dr = \Lambda, \tag{2.1}$$

where Λ is a constant during the decay process.

Loitsiansky remarks that the relation (2.1) and Eq. (1.3) are consistent with an analogy between the propagation of fu'^2, and the propagation of heat in a spherically-symmetrical five-dimensional field. In such a space the Laplacian operator has the form

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r},$$

so that the last term of Eq. (1.3) represents the effect of molecular conduction provided that fu'^2 is the analogue of temperature. The first term on the right side of (1.3) must be interpreted as the effect of convection, and (2.1) shows that this convective effect is such as to leave the total quantity of heat constant. On the basis of this analogy Loitsiansky describes the constant Λ as a measure of the total quantity of disturbance to the fluid, which is uniquely determined by the initial conditions of the turbulence.

It is not without interest to notice the behaviour of the general moment of the function $f(r)$. Thus

$$\frac{\partial}{\partial t}\left(u'^2 \cdot \int_0^\infty r^m f \, dr\right) = (4 - m)u'^3 \int_0^\infty k r^{m-1} \, dr + 2(m-1)(m-4)\nu u'^2 \int_0^\infty r^{m-2} f \, dr \tag{2.2}$$

provided that $m > 1$ and $(r^{m-1}f)_\infty = (r^m k)_\infty = 0$. When $m = 1$ or 0, special formulae are necessary, viz.

$$\frac{\partial}{\partial t}\left(u'^2 \cdot \int_0^\infty r f \, dr\right) = 3u'^3 \cdot \int_0^\infty k \, dr - 6\nu u'^2, \tag{2.3}$$

$$\frac{\partial}{\partial t}\left(u'^2 \cdot \int_0^\infty f \, dr\right) = 4u'^3 \cdot \int_0^\infty \frac{k}{r} \, dr + 8\nu u'^2 \int_0^\infty \frac{f'}{r} \, dr. \tag{2.4}$$

Now all the experimental evidence suggests that the function $f(r)$ is everywhere positive and monotonic decreasing, and the signs of the integrals containing f and f' can be predicted with safety. There is not very much data about $k(r)$ but measurements at the Cavendish Laboratory show it to be everywhere negative (or zero). It must certainly be negative for small values of r in order to give a positive contribution to the rate of change of mean square vorticity due to random extension of the vortex lines.[5] If the

[4] L. G. Loitsiansky, *Some basic laws of isotropic turbulent flow*, Central Aero- and Hydro-dynamic Institute, Moscow, Report No. 440, 1939; also N. A. C. A. Tech. Memo. 1079.

[5] G. K. Batchelor and A. A. Townsend, *Decay of vorticity in isotropic turbulence*, Proc. Roy. Soc. (A) **190**, 534-550 (1947).

signs of the integrals involving k are predicted on this basis, Eqs. (2.2), (2.3) and (2.4) show that

$$\left.\begin{aligned} \frac{\partial}{\partial t}\left(u'^2\cdot\int_0^\infty r^m f\,dr\right) &> 0 \quad \text{if} \quad m > 4, \\ &= 0 \quad \text{if} \quad m = 4, \\ &< 0 \quad \text{if} \quad m = 3, 2, 1, 0. \end{aligned}\right\} \tag{2.5}$$

These results have a bearing on the behaviour, during decay, of the spectrum function describing the spatial structure of the energy of the turbulence field. For if $u'^2\cdot F(\mu)\,d\mu$ is the amount of energy lying within a small range about the wave number μ the functions $f(r)$ and $F(\mu)/2(2\pi)^{1/2}$ are Fourier transforms,[6] and

$$F(\mu) = 4\int_0^\infty \cos 2\pi\mu r\cdot f(r)\,dr. \tag{2.6}$$

The relations (2.5) thus become, when m is even,

$$\frac{\partial}{\partial t}\left[u'^2\cdot\left(\frac{\partial^m F}{\partial\mu^m}\right)_{\mu=0}\right] \gtreqless 0 \quad \text{if} \quad m \gtreqless 4. \tag{2.7}$$

If the energy spectrum function is expanded in powers of μ^2, viz.

$$u'^2\cdot F(\mu) = u'^2\cdot F(0) + \frac{\mu^2}{2!}\left(\frac{\partial^2 u'^2 F}{\partial\mu^2}\right)_0 + \frac{\mu^4}{4!}\left(\frac{\partial^4 u'^2 F}{\partial\mu^4}\right)_0 + \cdots,$$

then the effect of decay is to decrease the coefficients of μ'^0 and μ^2, to leave the coefficient of μ^4 constant and to increase all other coefficients.

3. Self-preserving solutions at Reynolds number which are not large. By Reynolds number which are not large is meant a state of affairs in which λ is not small compared with other lengths associated with the function $f(r)$. Since the length λ is already present in the basic Eq. (1.3) (in view of the expression for du'^2/dt), it will be convenient mathematically to use λ as the scale factor L in any solution for $f(r)$ which preserves its shape over a range of r which includes small values of r. Several possibilities can be considered, the simplest of which is that the correlation functions preserve their shape for *all* values of r.

The hypothesis whose consequences are to be examined is that

$$f(r) \equiv f(\psi), \qquad k(r) \equiv k(\psi) \tag{3.1}$$

for all values of r, where $\psi = r/\lambda$. Equation (1.3) then becomes

$$-5f - \frac{\lambda}{2\nu}\frac{d\lambda}{dt}\psi\frac{df}{d\psi} = \frac{1}{2}\frac{u'\lambda}{\nu}\left(\frac{dk}{d\psi} + \frac{4k}{\psi}\right) + \left(\frac{d^2f}{d\psi^2} + \frac{4}{\psi}\frac{df}{d\psi}\right)$$

or, in terms of the number $R_\lambda = u'\lambda/\nu$,

$$\left(\frac{d^2f}{d\psi^2} + \frac{4}{\psi}\frac{df}{d\psi} + \frac{5\psi}{2}\frac{df}{d\psi} + 5f\right) + \frac{\lambda^2}{2\nu R_\lambda}\frac{dR_\lambda}{dt}\left(\psi\frac{df}{d\psi}\right) + \frac{1}{2}R_\lambda\left(\frac{dk}{d\psi} + \frac{4k}{\psi}\right) = 0. \tag{3.2}$$

[6] G. I. Taylor, *The spectrum of turbulence*, Proc. Roy. Soc. (A) **164**, 476-490 (1937).

Each of the expressions within circular brackets is a function of ψ only, while the coefficients outside the brackets depend only on t.

It is worth noting that the second coefficient, viz. $\lambda^2\, dR_\lambda/2\nu R_\lambda\, dt$, is a constant for a very general class of energy decay law. If u'^2 decays as some power of t, the energy equation (1.4) shows that the decay laws will be

$$u'^{-2} \sim t^n, \qquad \lambda^2 = \frac{10\nu}{n}\, t, \qquad R_\lambda^2 \sim t^{1-n} \tag{3.3}$$

and hence

$$\frac{\lambda^2}{2\nu R_\lambda} \frac{dR_\lambda}{dt} = \frac{5(1-n)}{2n}. \tag{3.4}$$

An exponential decay of u'^2 also makes this factor constant. When the second coefficient is constant, the first two groups of terms in (3.2) are functions of ψ only and the equation can only be satisfied by

$$R_\lambda = \text{constant}.$$

This leads to $n = 1$ provided the constant value of R_λ is not zero (in which case we should have $n > 1$, $t = \infty$). When the energy decay law is not such as to make $\lambda^2\, dR_\lambda/2\nu R_\lambda\, dt$ constant, there is another way in which Eq. (3.2) can be satisfied. This alternative method is suggested by the work of Sedov,[7] and will be discussed in section 5.

Now the law of energy decay is already fully determined by the assumption of self-preservation of the correlation function f for *all* values of r. For in this case the condition (2.1) becomes

$$u'^2\lambda^5 \int_0^\infty \psi^4 f(\psi)\, d\psi = \Lambda$$

provided that the integral converges, and thence $u'^2\lambda^5$ is constant during the decay. The energy equation then gives the decay laws as

$$u'^{-2} = A t^{5/2}, \qquad \lambda^2 = 4\nu t, \qquad R_\lambda^2 = \frac{4}{A\nu}\, t^{-3/2}, \tag{3.5}$$

where A is a constant and t is measured from the instant at which $1/u' = \lambda = 0$. This is a power law of energy decay (with $n = 5/2$) and as shown above, Eq. (3.2) can only be satisfied when R_λ is constant. Since R_λ also varies with t according to (3.5), the two requirements can only be consistent if

$$t = \infty, \qquad R_\lambda = 0.$$

Under these circumstances Eq. (3.2) becomes

$$\frac{d^2 f}{d\psi^2} + \frac{df}{d\psi}\left(\frac{4}{\psi} + \psi\right) + 5f = 0 \tag{3.6}$$

and the solution which makes $f = 1$ when $r = 0$ is

$$f(\psi) = e^{-\psi^2/2}. \tag{3.7}$$

[7] L. I. Sedov, *Decay of isotropic turbulent motions of an incompressible fluid*, C. R. Acad. Sci. U. R. S. S. 42, 116-119 (1944).

The requirement $R_\lambda = 0$ shows that the triple correlations have no influence on the double correlations under the conditions for which a completely self-preserving solution is possible.

Thus it has been shown that a solution in which the function f is completely self-preserving is only possible at large decay times and is described by (3.5) and (3.7). This suggests that we should examine conditions as $t \to \infty$ in order to see if such a self-preserving solution does in fact exist there. It is not difficult to see at once that the answer is likely to be in the affirmative. The correlation function $f(\psi)$ certainly has the same (parabolic) form for all values of t for $\psi \ll 1$, and the only alternative to an approach to a definite shape as $t \to \infty$ is an oscillation of the remaining part of the curve. Such an oscillation does not seem appropriate to the problem.

However, more definite evidence that $f(\psi)$ is independent of t when t is large can be obtained from the basic equation. For consider (2.3) in the form

$$\frac{\partial}{\partial t}\left(R_\lambda^2 \int_0^\infty \frac{r}{\lambda} f \, d\frac{r}{\lambda}\right) = \frac{3u'^2}{\nu} R_\lambda \int_0^\infty k \, d\frac{r}{\lambda} - 6\frac{u'^2}{\nu}.$$

All the measurements of $f(r)$ which have hitherto been made have shown it to be everywhere positive and we may safely assume the expression within brackets on the left side to be positive and, in view of the existence of a parabolic variation for $r/\lambda \ll 1$, non-zero, for all values of t. Also, as discussed in section 2, the evidence is that k is everywhere negative (or zero), so that

$$\frac{\partial}{\partial t}\left(R_\lambda^2 \int_0^\infty \frac{r}{\lambda} f \, d\frac{r}{\lambda}\right) \leq -6\frac{u'^2}{\nu} \tag{3.8}$$

Suppose now that when t is large, the decay of u' and λ conforms to the general laws (3.3). In the first place, n cannot be less than or equal to one, for (3.8) then requires $R_\lambda^2 \int_0^\infty r/\lambda \, f \, d\, r/\lambda$ to decrease indefinitely and to become negative as $t \to \infty$. Secondly, the inequality (3.8) requires the expression in brackets on the left side to vary as some power of t not greater than $1 - n$, whereas the factors in this expression show that it varies as some power of t not less than $1 - n$. Hence the power can only be $1 - n$ and $\int_0^\infty r/\lambda \, f \, d\, r/\lambda$ tends to a constant when t is large, which shows that f tends to become a function only of r/λ. Notice that the assumption $k \leq 0$ is over-sufficient; the deduction is only rendered invalid if $\int_0^\infty k \, d\, r/\lambda$ is positive and increases as some power of t not less than $(n - 1)/2$.

It is thus possible to make quite definite predictions about the turbulence when t is large. Making assumptions about f and k which are well supported by experiment, and assuming that the energy decay follows a power law (3.3), it can be shown that $f(r)$ preserves its shape when t is large. Then Loitsiansky's invariant relation shows that the decay laws must be as in (3.5). Finally the fundamental equation for f gives the solution (3.7). The dependence of these deductions on a decay law of the type specified by (3.3) when t is large does not seem likely to be critical.

4. Solutions obtained by neglecting the triple correlation. Two Russian authors, Loitsiansky[4] and Millionshtchikov,[8] have each discussed the solutions of Eq. (1.3) which are obtained when the term describing the effect of the triple velocity correlation

[8]M. Millionshtchikov, *Decay of homogeneous isotropic turbulence in a viscous incompressible fluid*, C. R. Acad. Sci. U. R. S. S. 22, 231-237 (1939).

is ignored. Their work is an extension of the "small Reynolds number" solution first put forward by v. Kármán and Howarth.[1] There is considerable indirect evidence for the belief that neglect of the triple correlations is only permissible at a late stage in the decay of turbulence so that the resulting solutions ought to be compared with that deduced in the previous section for t large.

The equation to be solved is

$$\frac{\partial u'^2 f}{\partial t} = 2\nu\left(\frac{\partial^2 u'^2 f}{\partial r^2} + \frac{4}{r}\frac{\partial u'^2 f}{\partial r}\right), \tag{4.1}$$

which is similar to the equation for the propagation of heat in a spherically-symmetrical five-dimensional field in which there is no convection. Using this analogy the solution is known to be

$$u'^2 f = \frac{1}{(8\pi\nu t)^{5/2}} \int\!\!\int\!\!\int\!\!\int\!\!\int_{-\infty}^{\infty} F(s, 0) e^{-\rho^2/8\nu t}\, dx_1\, dx_2\, dx_3\, dx_4\, dx_5 \tag{4.2}$$

where

$$F(r, t) = u'^2 f,$$

$$\rho^2 = \sum_{n=1}^{5} (\xi_n - x_n)^2, \qquad r^2 = \sum_1^5 \xi_n^2, \qquad s^2 = \sum_1^5 x_n^2.$$

The law of energy decay is obtained by putting $r = 0$ in (4.2), i.e.

$$u'^2 = \frac{1}{(8\pi\nu t)^{5/2}} \int\!\!\int\!\!\int\!\!\int\!\!\int_{-\infty}^{\infty} F(s, 0)\, e^{-s^2/8\nu t}\, dx_1\, dx_2\, dx_3\, dx_4\, dx_5$$

$$= \frac{1}{48(2\pi)^{1/2}(\nu t)^{5/2}} \int_0^{\infty} F(s, 0)\, e^{-s^2/8\nu t} s^4\, ds. \tag{4.3}$$

Thus, in general, the solution depends upon, and is uniquely determined by, the function $F(r, 0)$.

If the initial state of the turbulence is such that

$$\left.\begin{aligned} F(r, 0) &= 0, \quad \text{when} \quad r > 0, \\ &= \infty, \quad \text{when} \quad r = 0, \\ \int_0^{\infty} F(r, 0) r^4\, dr &= \Lambda \end{aligned}\right\} \tag{4.4}$$

where Λ is finite, then the integrals can be evaluated giving

$$u'^2 = \frac{\Lambda}{48(2\pi)^{1/2}} (\nu t)^{-5/2}, \tag{4.5}$$

$$f(r, t) = e^{-r^2/8\nu t} = e^{-r^2/2\lambda^2}. \tag{4.6}$$

Loitsiansky describes these initial conditions as referring to a "point source of strength Λ" since the analogous problem in the five-dimensional field is simply the spread of heat from an initial point source. Millionshtchikov has also obtained this special solution and remarks that it describes the turbulence which exists subsequent to an initial random distribution of concentrated line eddies, provided that the effect of triple correlations is ignored. Since the triple velocity correlations cannot be neglected in the early stages of the decay when u' is not small compared with the characteristic velocity of the turbulence-producing device, Millionshtchikov's interpretation of the initial conditions (4.4) cannot be regarded as having physical reality.

v. Kármán and Howarth considered the particular set of solutions of (4.1) which are functions of $r/(\nu t)^{1/2}$ only, i.e. a self-preserving solution was assumed. The solution (4.6) is the only solution of this kind if certain conditions concerning the behaviour of $f(r)$ are accepted. v. Kármán and Howarth gave a family of self-preserving solutions of (4.1) with the quantity $\alpha, = \nu t/\lambda^2$, as parameter, viz.

$$f(x) = 2^{15/4}\chi^{-5/2}e^{-\chi^2/16}M_{10\alpha-5/4,3/4}(\chi^2/8), \tag{4.7}$$

where $\chi = r/(\nu t)^{1/2}$, and $M_{k,m}(z)$ is the same solution of the confluent hypergeometric equation as that defined by Whittaker and Watson (*Modern analysis*, 1927, p. 337) and denoted by this symbol. The solution (4.6) corresponds to the particular value $\alpha = \frac{1}{4}$. Now when $\alpha < \frac{1}{4}$, the expression (4.7) is proportional to $\chi^{-20\alpha}$ when χ is large. If the restriction that the various moments of $f(r)$ should all be finite is accepted on intuitive physical grounds, then certain values of α can be rejected. In particular, if the fourth moment $\int_0^\infty r^4 f\,dr$ (which occurs in the invariant of Sec. 2) is required to be finite then all values of α less than $\frac{1}{4}$ must be rejected.

On the other hand, when $\alpha > \frac{1}{4}$, the expression (4.7) becomes negative for certain values of r since it has the expansion

$$f(\chi) = e^{-\chi^2/8}\left\{1 + \sum_{n=1}^{\infty} \frac{(2.5 - 10\alpha)(3.5 - 10\alpha)\cdots(2.5 - 10\alpha + n - 1)}{n!\,(2.5)(3.5)\cdots(2.5 + n - 1)}\left(\frac{\chi^2}{8}\right)^n\right\} \tag{4.8}$$

valid for all finite values of χ (Whittaker and Watson, p. 337). (Note that although the fourth moment $\int_0^\infty r^4 f\,dr$ converges when $\alpha > \frac{1}{4}$, the invariant relation of section 2 cannot be employed to obtain a definite value for α and thence a definite decay law, because the solution makes the fourth moment vanish and Λ is constant whatever the energy decay law). Negative values of f have never been measured, so that there are some physical grounds for rejecting values of α greater than $\frac{1}{4}$. Thus when these restrictions are applied, v. Kármán and Howarth's family of self-preserving solutions reduces to the single solution (4.6). Both restrictions are of course implicit in Loitsiansky's analogy with the propagation of heat in a spherically-symmetrical five-dimensional field.

Let us return now to the general solution (4.2), with its associated decay law (4.3), which satisfies the Eq. (4.1) obtained by neglecting the triple correlation term. When t is so large that $\nu t > s^2$ for all values of s for which $F(s, 0)$ is large enough to contribute to the integrals, the integrals in (4.3) and (4.2) simplify. Thus when $t \to \infty$,

$$u'^2 \to \frac{1}{48(2\pi)^{1/2}(\nu t)^{5/2}} \int_0^\infty F(s, 0)s^4\,ds$$

and

$$u'^2 f \to \frac{1}{48(2\pi)^{1/2}(\nu t)^{5/2}} \int_0^\infty F(s, 0)\, e^{-r^2/8\nu t} s^4\, ds,$$

i.e., $\quad f \to e^{-r^2/8\nu t}.$

These limiting forms show that if the correlation $f(r, t)$ and turbulence intensity u'^2 are calculated on the assumption that the triple correlation is without effect, the solutions obtained tend to the forms (4.5) and (4.6) as $t \to \infty$ whatever the choice of conditions at $t = 0$. The common limiting form is a self-preserving solution, so that there is here further support for the contention of the previous section that a self-preserving solution does exist when t is large, and that it is given by Eqs. (4.5) and (4.6).

In a later work[9] Millionshtchikov made an attempt to determine the effect of triple correlations at decay times which are not large, using a reiterative method. However his attempt is based upon the existence of the solution (4.6) as a first approximation and leads only to solutions which have the same self-preserving character. Since a self-preserving solution has been shown to be possible only when t is large, it is questionable whether Millionshtchikov's approximate solutions at decay times which are not large have any significance.*

5. Partially self-preserving solutions at Reynolds number which are not large. It has been seen that a correlation function which is completely self-preserving can only occur when the decay time is large. On the other hand, it is known that some measure of self-preservation exists when t is not large. The function f is always parabolic near $r = 0$. Recent experiments[5] have indicated that the expansion of f in powers of r/λ as far as the term of fourth degree, and of k as far as the term of third degree, are independent of t at decay times which are not large. This suggests that we should explore the consequences of assuming partially self-preserving solutions for the correlation functions. We therefore write

$$f(r) \equiv f(\psi), \qquad k(\psi) \equiv k(\psi), \qquad \psi = \frac{r}{\lambda},$$

for a range of values of r, $0 \leq r < l$, where l is an unknown length. From the above evidence l must be at least as great as the maximum value of r for which a fourth degree polynominal gives a good representation of $f(r)$. The fundamental Eq. (1.3) again reduces to the form (3.2) for the restricted range of r, viz.

$$\left(f'' + \frac{4}{\psi} f' + \frac{5\psi}{2} f' + 5f\right) + \frac{1}{2} \frac{\lambda^2}{\nu R_\lambda} \frac{dR_\lambda}{dt} (\psi f') + \frac{1}{2} R_\lambda \left(k' + \frac{4}{\psi} k\right) = 0. \qquad (3.2)$$

Since the hypothesis leaves arbitrary the behaviour of the correlation functions at large values of r, we cannot make use of the invariant relation of Sec. 2. The energy

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[9] M. Millionshtchikov, *On the theory of homogeneous isotropic turbulence*, C. R. Acad. Sci. U. R. S. S. **32**, 615-621 (1941).

*A further criticism of Millionshtchikov's work is that he has omitted to take into consideration the correlation between two velocity components and the pressure when determining the relation between triple and quadruple velocity correlations. His idea of determining quadruple velocity correlations from the approximation that the relation between double and quadruple correlations is as it is for a Gaussian distribution of the sums of velocity components at two points seems to be useful and to warrant further exploitation.

decay laws are therefore not prescribed as they were in the case of completely self-preserving solutions and there will be a degree of indeterminacy in the deductions.

It was mentioned in Sec. 3 that there are in general two methods of satisfying Eq. (3.2) for non-zero ranges of ψ and t. According to the first method, in which the energy decay follows a power law, R_λ is constant and for non-zero values of this constant we have $n = 1$, i.e.

$$u'^{-2} = Bt, \qquad \lambda^2 = 10\nu t, \qquad R_\lambda = \frac{10}{B\nu} \tag{5.1}$$

where B is a constant. The functions $f(\psi)$ and $k(\psi)$ are in this case connected by the equation

$$f'' + f'\left(\frac{4}{\psi} + \frac{5\psi}{2}\right) + 5f + \frac{1}{2}R_\lambda\left(k' + \frac{4}{\psi}k\right) = 0 \tag{5.2}$$

provided $0 \leq r < l$. The relations (5.1) and (5.2) become identical with Dryden's deductions[10] from the postulate of self-preservation when his scale factor L is replaced by the length used in the present analysis, viz. λ; Dryden took Eq. (5.2) to be valid for all values of r but we have already seen that the assumption of complete self-preservation leads to a quite different set of results.

It does not seem possible to check Eq. (5.2) relating f and k, since measurements of k are difficult to make and no results have yet been published. However, measurements of u' at different stages of decay have frequently been made, and the validity of (5.1) can be assessed. The evidence for the law of energy decay has been discussed by Dryden.[11] The data from different sources are not wholly consistent, but inasmuch as any one law does describe the experimental relations, it is $u'^{-2} \sim t^n$, where n lies between one and two. In more recent experiments at the Cavendish Laboratory, Cambridge,[5] the decay of λ has been measured simultaneously with that of u' and the energy equation has in this case provided a check on the consistency of the two sets of measurements. It was in fact found that the relations (5.1) were obeyed to a quite high degree of accuracy for the decay range $40\ M/U < t < 120\ M/U$, where M and U are the periodic length and velocity associated with the turbulence-producing grid and were varied over the range $5.5 \times 10^3 < UM/\nu < 2.2 \times 10^4$. Under these same conditions it was found, as mentioned above, that $f(r)$ and $k(r)$ were self-preserving during decay at least as far as the terms in r^4 and r^3 respectively. The above experiments are therefore quite consistent with the postulate of limited self-preservation and with the deductions obtained by using the first method of satisfying Eq. (3.2). A point left open is the value of l, which presumably can be determined by the region of validity of the relation (5.2).

According to the second method of satisfying Eq. (3.2) (see Sedov[7]), the coefficients of bracketed terms therein are related by

$$\frac{\lambda^2}{\nu R_\lambda}\frac{dR_\lambda}{dt} = aR_\lambda + b \tag{5.3}$$

where a and b are constants. For solutions of this equation which do not make R_λ constant, f and k must then satisfy the equations

[10] H. Dryden, *Isotropic turbulence in theory and experiment*, v. Kármán Anniversary Volume on Applied Mechanics, 1941, pp. 85-102.

[11] H. Dryden, *A review of the statistical theory of turbulence*, Q. Appl. Math. 1, 7-42 (1943).

$$f'' + \frac{4}{\psi} f' + \left(\frac{5+b}{2}\right)\psi f + 5f = 0, \tag{5.4}$$

$$k' + \frac{4}{\psi} k = -a\psi f', \tag{5.5}$$

provided $0 \leq r < l$. Apart from the constants a and b, this method of solution provides unique determinations of the correlations f and k within the restricted range of r.*

Consider the decay law described by (5.3). This equation is to be solved with the aid of the energy equation in the form

$$\frac{2\lambda^2}{\nu R_\lambda}\frac{dR_\lambda}{dt} = \frac{d\lambda^2/\nu}{dt} - 10. \tag{5.6}$$

The solution cannot be obtained explicitly, and is given by

$$\frac{\lambda^2}{\nu} = \frac{R_\lambda^2}{K(a + b/R_\lambda)^{10/b}} = \frac{\nu R_\lambda^2}{u'^2} \tag{5.7}$$

where K is a constant of integration, and

$$\frac{dR_\lambda}{dt} = K\left(a + \frac{b}{R_\lambda}\right)^{1+10/b}. \tag{5.8}$$

It will be seen in a moment that a is necessarily negative and it is then evident from (5.7) that b must be positive if u' is to become large for some values of R_λ. The nature of the decay relations specified by (5.3) is as indicated in Fig. 1. The variation of u'

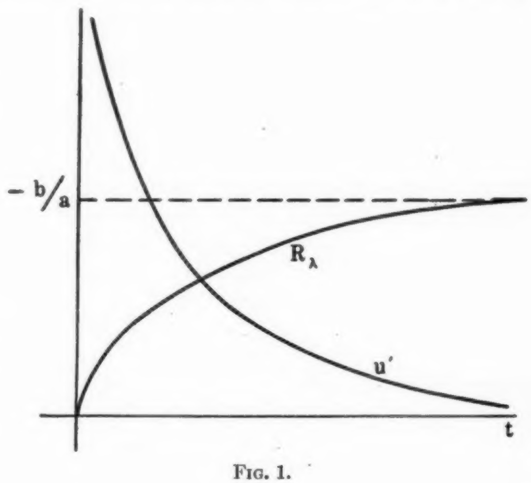

Fig. 1.

has the general features of measured energy decay curves. R_λ increases with decay time and asymptotes to a constant value $-b/a$. It follows that u'^{-2} and λ^2 asymptote to the values $(-a/b)\ 10t/\nu$ and $10\ \nu t$ respectively as t increases.

*This might be regarded as a hint that the solution is physically impossible. However I have not been able to find any definite anomalies.

It might be argued that this method of satisfying Eq. (3.2) has thus far led to predictions which are not inconsistent with experiment, since we cannot be sure that the asymptotic variation (R_λ = constant) does not occur over the range of decay times used in the experiments mentioned earlier. But the Eqs. (5.4) and (5.5) determining the functions $f(\psi)$ and $k(\psi)$ make possible a further comparison with experiment. There is available sufficient experimental evidence to determine the value of the constant a in (5.5). It has been found[5] that when r is sufficiently small, the function $k(r)$ has the form

$$k(r) = -\frac{1}{6}S\left(\frac{r}{\lambda}\right)^3, \tag{5.9}$$

where S is an absolute constant (of value about 0.39) for the ranges of decay times and mesh Reynolds numbers used in the experiments. There are also theoretical reasons,[12] derived from Kolmogoroff's theory of locally isotropic turbulence,[13] for believing that S is an absolute constant whenever the Reynolds number is sufficiently high. Thus comparison of the coefficients of powers of r^2 in (5.5) shows

$$a = -\frac{7}{6}S. \tag{5.10}$$

Then if R'_λ be written for the asymptotic value of the Reynolds number R_λ, $= u'\lambda/\nu$, the value of b is

$$b = -aR'_\lambda = \frac{7}{6}SR'_\lambda . \tag{5.11}$$

Changing the variable of (5.4) to χ, $= \psi/(\alpha)^{1/2}$, where

$$\alpha = \frac{1}{10 + 2b} = \frac{1}{10 + 7SR'_\lambda/3},$$

the equation for f becomes

$$f'' + f'\left(\frac{4}{\chi} + \frac{\chi}{4}\right) + 5\alpha f = 0. \tag{5.12}$$

v. Kármán and Howarth[1] have pointed out, in a slightly different context, that the solution of this equation is related to the hypergeometric function and that the solution which satisfies $f = 1$ and $f' = 0$ when $r = 0$, can be written

$$f(\chi) = \frac{\Gamma(5/2)}{\Gamma(10\alpha)\Gamma(5/2 - 10\alpha)}\int_0^1 \tau^{10\alpha-1}(1 - \tau)^{3/2-10\alpha}e^{-(\chi^2\tau)/8}\,d\tau \tag{5.13}$$

when, as is here the case, $\alpha < \frac{1}{4}$. Knowing the value to which $u'\lambda/\nu$ tends according to this method of satisfying the requirement of limited self-preservation of the correlation functions, it is thus possible to determine f. Figure 2 shows not f, but the related double correlation function g, which, in the notation of Sec. 1, is given by

$$u'^2 g(r) = \overline{(u_i u'_j)}_{\substack{i=j\\ \xi_i=0}} .$$

[12] G. K. Batchelor, *Kolmogoroff's theory of locally isotropic turbulence*, Proc. Camb. Phil. Soc. (to be published).

[13] A. N. Kolmogoroff, *The local structure of turbulence in an incompressible viscous fluid for very large Reynolds numbers*, C. R. Acad. Sci. U. R. S. S. **30**, 301-5 (1941) and **32**, 16-18 (1941).

$g(r)$ is easier to measure in practice, and is related to $f(r)$ by the continuity equation

$$g(r) = f(r) + \frac{r}{2}\frac{\partial f(r)}{\partial r}. \tag{5.14}$$

g has been calculated from (5.13) and (5.14), in part by numerical integration and in part by the use of an asymptotic expansion of the integral, for a number of values of R'_λ comparable with those found in the experiments at Cambridge. The largest value, viz. 44, is the value of $u'\lambda/\nu$ measured at intermediate decay times with a mesh of circular rods at 1″ spacing and a wind speed of 42 ft/sec.

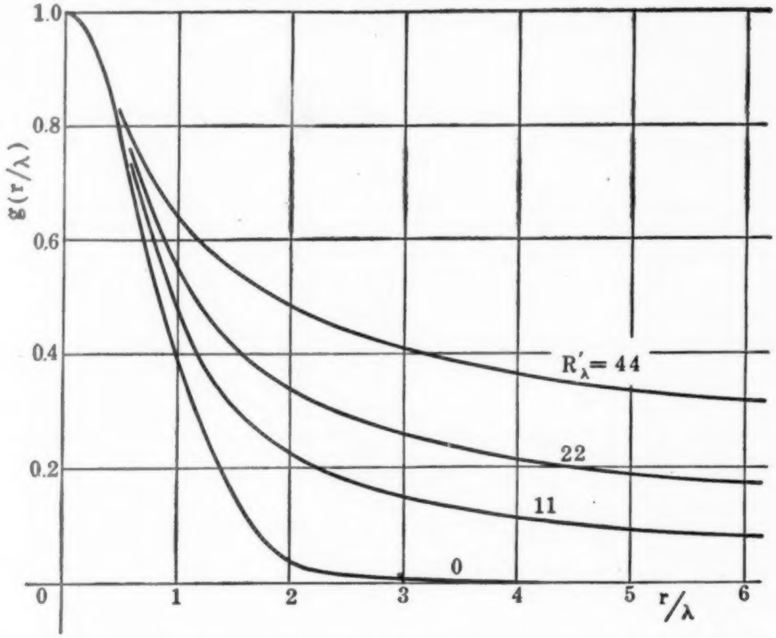

FIG. 2. Solutions of $d^2f/d\psi^2 + (4/\psi + \frac{1}{2}(5 + b)\psi)\, df/d\psi + 5f = 0$ and $g = f + \frac{1}{2}\psi\, df/d\psi$, where $\psi = r/\lambda$, $b = 7SR'_\lambda/6$, $S = 0.39$.

Mr. A. A. Townsend and the author hope that it will soon be possible to present measurements of $g(r)$ during decay under different conditions in order to determine the validity of the solution (5.13). Assuming that the measurements show $u'\lambda/\nu$ to be constant at decay times which are not too large and that self-preservation of the function $f(r)$ during decay occurs over a limited range $0 \leq r < l$, then agreement between experiment and the curves of Fig. 2 over this same range* of values of r would show that

*A hint concerning the possible range of validity of the curves in Fig. 2 is provided by the requirement, deduced from (5.14), that the *complete* function $g(r)$ obeys the relation

$$\int_0^\infty rg(r)\, dr = 0.$$

the second method of satisfying Eq. (3.2) is here valid. Necessary consequences are that k is given by (5.5), i.e.

$$k(\psi) = \frac{7}{6} S\psi^{-4} \int_0^\psi \psi^5 \frac{df}{d\psi} d\psi \qquad (0 \le r < l) \tag{5.15}$$

where f is given by the expression (5.13), and that at decay times which are sufficiently small the decay of u' is such that $u'\lambda/\nu$ increases from a small value to its asymptotic value R'_λ. On the other hand, lack of confirmation of the curves of Fig. 2 would show that the first method of satisfying Eq. (3.2) is valid, i.e. that $u'\lambda/\nu$ is constant over the whole of the decay range for which limited self-preservation holds and that k is determined by (5.2) as

$$R_\lambda k(\psi) = -2\left(\frac{df}{d\psi} + \psi f\right) - 3\psi^{-4} \int_0^\psi \psi^5 \frac{df}{d\psi} d\psi. \tag{5.16}$$

The function $f(\psi)$ is not prescribed by this method of satisfying (3.2), except in the particular case mentioned below.

The case in which R_λ, or R'_λ in the second method, is very small (but constant) is particularly interesting, for the two methods of satisfying (3.2) then lead to identical approximate equations for $f(\psi)$, viz.

$$f'' + f'\left(\frac{4}{\psi} + \frac{5\psi}{2}\right) + 5f = 0. \tag{5.17}$$

The solution is shown in Fig. 2 and it can be predicted that the correlation function $f(r)$ will approximate to this shape at low Reynolds numbers of turbulence over the range of values of r for which this same correlation preserves its form during decay.

6. Solutions at high Reynolds number. (*a*) *No assumption of self-preservation.* Using the energy Eq. (1.4), the basic Eq. (1.3) can be written

$$u'\left(k' + \frac{4k}{r}\right) = \frac{\partial f}{\partial t} - \nu\left(\frac{10f}{\lambda^2} + f'' + \frac{4f'}{r}\right). \tag{6.1}$$

We compare the orders of magnitudes of the two terms

$$\frac{10f}{\lambda^2} \qquad \text{and} \qquad f'' + \frac{4f'}{r}.$$

When r is sufficiently small for f to be represented by the parabola $1 - r^2/2\lambda^2$, these terms are of equal order of magnitude, however small λ may be. Now when the Reynolds number of the turbulence is increased observation shows that the correlation curve does not change appreciably outside the parabolic region, the change being confined to a diminution in the value of λ. The first of the two terms will therefore become relatively large at high Reynolds numbers for values of r lying beyond the parabolic region, and the approximate form of (6.1) is

$$u'\left(k' + \frac{4k}{r}\right) = \frac{\partial f}{\partial t} - \frac{10\nu f}{\lambda^2}. \tag{6.2}$$

It is possible to integrate Eq. (6.2) if r is further restricted to be sufficiently small for the approximation $f = 1$ to be valid. In this case,

$$k' + \frac{4k}{r} = -\frac{10\nu}{\lambda^2 u'} \qquad = -\frac{10}{\lambda R_\lambda} \tag{6.3}$$

and the solution which makes $k(r)$ vanish at the origin is

$$k(r) = -\frac{2}{R_\lambda}\frac{r}{\lambda}. \tag{6.4}$$

The range in which (6.4) can be expected to apply is rather restricted. In the first place, the Reynolds number must be large enough for (6.2) to apply. r must be large enough to lie outside the parabolic region of $f(r)$, but in view of the requirement of high Reynolds number this is not an important limitation. More important is the requirement that r should be small enough for the value of f to be close to unity. The meaning of (6.4) is evidently that at large Reynolds numbers, the region near the origin in which $k(r)$ is cubic becomes small and the curve tends to a straight line with a slope which is determined by the turbulence Reynolds number R_λ. This discussion is substantially that already given by Kolmogoroff.[13]

(b) *Self-preserving solutions.* Under the assumed conditions of high Reynolds number, the dissipation length parameter λ is small compared with other lengths associated with the turbulence. In the limit, λ is zero and the tangent to the correlation function $f(r)$ at $r = 0$ is not horizontal, but makes an angle with the abscissa which is probably 90°. In discussing self-preserving solutions at high Reynolds numbers it is no longer possible to use λ as the reference length and some other length L associated with $f(r)$ will be employed. We specifically permit λ/L to vary during decay, since we should otherwise merely repeat the analysis of Sec. 3.

The present hypothesis is thus

$$f(r) \equiv f(\chi), \qquad k(r) \equiv k(\chi), \qquad \chi = \frac{r}{L} \tag{6.5}$$

for a range of values of r to be specified later. The reference length L must of course be defined by values of the function $f(r)$ within the range of r for which (6.5) is to hold. The basic equation (1.3) becomes

$$-5\frac{L^2}{\lambda^2}f - \frac{L}{2\nu}\frac{dL}{dt}\chi\frac{df}{d\chi} = \frac{1}{2}R_L\left(\frac{dk}{d\chi} + \frac{4k}{\chi}\right) + \left(\frac{d^2f}{d\chi^2} + \frac{4}{\chi}\frac{df}{d\chi}\right) \tag{6.6}$$

where $R_L = u'L/\nu$.

L/λ is large under the assumed conditions, and the term $(d^2f/d\chi^2 + 4/\chi\, df/d\chi)$ should be neglected by comparison with the term $5fL^2/\lambda^2$ in order to be consistent with the assumption that the non-self-preserving parabolic region near the origin is infinitesimal in extent. Hence (6.6) becomes

$$10\frac{L^2}{\lambda^2}(f) + \frac{L}{\nu}\frac{dL}{dt}(\chi f') + R_L\left(k' + \frac{4k}{\chi}\right) = 0. \tag{6.7}$$

Equation (6.7) will be satisfied if the three coefficients, which are functions only of t, are proportional, i.e. if

$$R_L = A\, L^2/\lambda^2, \tag{6.8}$$

$$\frac{L}{\nu}\frac{dL}{dt} = BR_L\,. \tag{6.9}$$

These are the equations derived by v. Kármán[1] in a discussion of self-preserving solutions at high Reynolds number. The equation connecting f and k becomes

$$10f + AB\chi f' + A\left(k' + \frac{4}{\chi}k\right) = 0. \tag{6.10}$$

As before, it is also possible to satisfy Eq. (6.7) in the manner suggested by Sedov,[7] but in this case there are reasons for rejecting this alternative. The appropriate relation between the coefficients is

$$10\frac{L^2}{\lambda^2} = a\frac{L}{\nu}\frac{dL}{dt} + bR_L\,, \tag{6.11}$$

where a and b are constants. For solutions of this equation which do not make $L\,dL/\nu\,dt$ proportional to R_L, the functions f and k must then satisfy

$$af + \chi f' = 0, \tag{6.12}$$

$$bf + k' + \frac{4}{\chi}k = 0. \tag{6.13}$$

However Eq. (6.12) has no solutions such that $f = 1$ when $\psi = 0$ and the method of satisfying (6.7) must be rejected.

v. Kármán's Eqs. (6.8) and (6.9), and Eq. (6.10), therefore appear as necessary consequences of self-preservation at high Reynolds numbers. The meaning of (6.8) becomes clear when we substitute for λ^2 in the energy equation to obtain

$$\frac{du'^2}{dt} = -\frac{10}{A}\frac{u'^3}{L}.$$

This equation shows that the quantities u' and L may be considered as characteristic of the whole turbulence in a calculation of the work done against Reynolds stresses, as indeed is to be expected when the correlation functions preserve their shape during decay. Perhaps less to be expected is that this expression for the energy decay remains valid (as will be seen later) when the correlation functions are only partially self-preserving.

When r is small and f may be replaced by unity, Eq. (6.10) becomes

$$\frac{dk}{d\chi} + \frac{4}{\chi}k = -\frac{10}{A} = -10\frac{L^2}{\lambda^2 R_L}$$

which is identical with Eq. (6.3) derived without any assumption of self-preservation. Clearly the general solution (6.4), viz.

$$k(r) = -\frac{2}{R_\lambda}\frac{r}{\lambda},$$

is self-preserving with L as the unit of length when the decay law (6.8) holds.

The decay equations (6.8) and (6.9) contain three variables, L, λ, u' and may be solved with the help of the energy equation (1.4). The differential equation for L is

$$L\frac{d^2L}{dt^2} = -\frac{5}{AB}\left(\frac{dL}{dt}\right)^2, \tag{6.14}$$

and has the solution*

$$L = L_0\left(\frac{t}{t_0}\right)^{(AB)/(5+AB)}. \tag{6.15}$$

Then from (6.8) and (6.9)

$$\frac{1}{u'} = \frac{(5 + AB)t_0}{AL_0}\left(\frac{t}{t_0}\right)^{5/(5+AB)} = \frac{1}{u_0'}\left(\frac{t}{t_0'}\right)^{5/(5+AB)}, \tag{6.16}$$

$$\lambda^2 = \nu(5 + AB)t, \tag{6.17}$$

where L_0 and u_0' are the values of L and u' at $t = t_0$.

It is at this stage that we must be more specific about the range of values of r over which self-preservation is to be postulated. If the correlation functions are completely self-preserving, Loitsiansky's invariant relation (2.1) shows that

$$u'^2L^5 = \text{constant}, \tag{6.18}$$

and we must have

$$AB = 2,$$

as has been pointed out by Kolmogoroff.[14] The decay laws then become

$$\frac{L}{L_0} = \left(\frac{t}{t_0}\right)^{2/7}, \qquad \frac{u_0'}{u'} = \left(\frac{t}{t_0}\right)^{5/7}, \qquad \lambda^2 = 7\nu t. \tag{6.19}$$

The corresponding relation between f and k is, from (6.10),

$$10f + 2\chi f' + A\left(k' + \frac{4}{\chi}k\right) = 0 \tag{6.20}$$

which can be integrated to give

$$k = -\frac{2}{A}\chi f = -\left(\frac{2L_0}{7t_0u_0'}\right)\frac{r}{L}f. \tag{6.21}$$

At the present time, we have not got sufficient evidence at high Reynolds numbers of turbulence to determine the validity of the decay laws (6.19) and the correlation relation (6.21).

If, on the other hand, the hypothesis of partial self-preservation only is made, the value of AB remains arbitrary and deductions about the decay laws finish at the relations (6.15)–(6.17). We can consider one or two consequences of particular values of AB. The assumption of large Reynolds numbers entered the analysis via the postulate that L/λ is large. Since (6.15) and (6.17) show that

$$\frac{L}{\lambda} = \frac{L_0}{[(5 + AB)\nu t_0]^{1/2}}\left(\frac{t}{t_0}\right)^{(AB-5)/(2(5+AB))}, \tag{6.22}$$

the subsequent behaviour of L/λ depends critically on the sign of $AB - 5$.

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*v. Kármán[1] has written the exponent of the solution in error as $5/(5 + AB)$.

[14]A. N. Kolmogoroff, *On degeneration of isotropic turbulence in an incompressible viscous liquid*, C. R. Acad. Sci. U. R. S. S. **31**, 538-540 (1941).

If $AB > 5$, L/λ (and also $u'L/\nu$) increases and the assumed state of affairs applies with ever-increasing accuracy. A state of turbulence in which the Reynolds number $u'L/\nu$ increases indefinitely—although the energy decreases—does not seem possible, and an argument of the kind used in Sec. 3 does in fact confirm this impression. Equation (2.3) can be written

$$\frac{\partial}{\partial t}\left(R_L^2 \int_0^\infty \frac{r}{L} f\, d\frac{r}{L}\right) = \frac{3u'^2}{\nu} R_L \int_0^\infty k\, d\frac{r}{L} - \frac{6u'^2}{\nu}. \tag{6.23}$$

Suppose that when $t \to \infty$, we may write

$$\int_0^\infty \frac{r}{L} f\, d\frac{r}{L} \sim t^\alpha, \qquad \int_0^\infty k\, d\frac{r}{L} \sim t^\beta$$

and the decay laws (6.17)–(6.19) are valid. Replacing the terms of equation (6.23) by their orders of magnitude in t,

$$0(t^{(2AB-10)/(5+AB)+\alpha-1}) \sim 0(t^{(AB-15)/(5+AB)+\beta}) + 0(t^{(-10)/(5+AB)}).$$

In the present analysis the viscous term (i.e. the last on the right side) is assumed to be without effect, so that the remaining terms must be of equal order, i.e. $\alpha = \beta$. Moreover, with the assumption made previously that $\int_0^\infty k\, dr < 0$, the term on the left of (6.23) must be negative and consequently

$$\alpha < \frac{10 - 2AB}{5 + AB}.$$

The first term on the right side of (6.23) thus varies as a power of t which is less than (-1), and the viscous term will only be of smaller order if $AB < 5$. Since L/λ increases indefinitely with t when $AB > 5$, we have here an inconsistency which prohibits such values of AB except at small values of t.

The case $AB = 5$ leads to the decay laws

$$u'^{-2} \sim t, \qquad L^2 \sim \lambda^2 = 10\, \nu t.$$

Since L/λ is constant, there is no tendency for the approximation on which the solution is based to become invalid as t increases. Nevertheless such decay laws cannot persist indefinitely. When t is large, the first term on the right side of (6.23) must vary as some power of t not less than (-1) if the viscous term is not to become dominant; however such a variation with t requires $\int_0^\infty (r/L) f d(r/L)$ to approach $-\infty$, which is impossible.

Finally, when $AB < 5$, L/λ diminishes as t increases so that there comes a time when the postulate of high Reynolds number ceases to be valid; the case $AB = 2$ deduced from complete self-preservation is in this category. The system is here naturally unstable, whereas if the case $AB \geq 5$ occurs at all some secondary factor disturbs the system before t becomes large. Thus none of the regimes deduced for large Reynolds number are possible for indefinitely large times of decay.

7. Conclusion. The various hypotheses discussed in the last three sections, and the deductions made from them, are summarized in the following table.

Hypotheses or conditions	Reynolds number not large	Large Reynolds number
—	—	$k(r) = -\frac{2}{R_\lambda}\frac{r}{\lambda}$ for r so small that $f(r) \approx 1$
$t \to \infty$	$u'^{-2} \sim t^{5/2}, \quad \lambda^2 = 4\nu t,$ $f(\psi) = e^{-\psi^2/2}$ where $\psi = \frac{r}{\lambda}$	—
complete self-preservation of $f(r)$ and $k(r)$ during decay	Only possible when $t \to \infty$; see entry above.	$u'^{-2} \sim t^{10/7}, \quad L^2 \sim t^{4/7}, \quad \lambda^2 = 7\nu t$ $k\left(\frac{r}{L}\right) = -\frac{2}{7}\left(\frac{L_0}{t_0 u_0'}\right)\frac{r}{L} f\left(\frac{r}{L}\right)$
self-preservation of $f(r)$ and $k(r)$ for $0 \leq r < l$	Either (1) $u'^{-2} \sim t, \quad \lambda^2 = 10\nu t$ $f'' + f'\left(\frac{4}{\psi} + \frac{5\psi}{2}\right) + 5f + \frac{1}{2}R_\lambda\left(k' + \frac{4}{\psi}k\right) = 0 \quad (0 \leq r < l)$ or (2) $u'^2 = K\nu\left(a + \frac{b}{R_\lambda}\right)^{10/b},$ $t = \frac{1}{K}\int\left(a + \frac{b}{R_\lambda}\right)^{-1-10/b} dR_\lambda$ $f'' + f'\left(\frac{4}{\psi} + \frac{\psi}{4}\right) + \left(\frac{5}{10 + 0.91 R_\lambda'}\right)f = 0,$ $k(\psi) = \frac{0.46}{\psi^4}\int_0^\psi \psi^5 f' \, d\psi,$ where $R_\lambda' = -\frac{b}{a}, \quad 0 \leq r < l.$ In either case, $f'' + f'\left(\frac{4}{\psi} + \frac{5\psi}{2}\right) + 5f = 0 \quad (0 \leq r < l)$ when R_λ or $R_\lambda' = 0$	$u'^{-2} \sim t^{10/(5+AB)}, \; L^2 \sim t^{2AB/(5+AB)},$ $\lambda^2 = (5 + AB)\nu t$ $10f + AB\chi f' + A\left(k' + \frac{4}{\chi}k\right) = 0$ where $\chi = \frac{r}{L}, \quad 0 \leq r < l.$

In conclusion, it should be noted that even if the deductions listed in the table are found to be in agreement with the appropriate measurements, the analysis of this paper does not by any means constitute a solution to the problem of the decay of isotropic turbulence. All that can be inferred from agreement with experiment is that the decay process takes place in a certain manner, for example, with self-preservation of the correlation functions for a certain range of r and all that this implies. (The deductions about the turbulence as $t \to \infty$, or when r is small and the Reynolds number is large, are exceptions to this statement since no hypotheses were required in these cases). In other words, the hypotheses of the above table are mathematical in origin and, so far, have no physical raison d'être. The task now is to find physical reasons why the decay process for t not large takes place as it does; indeed this might be said to have always been the chief problem of research on isotropic turbulence. Kolmogoroff's similarity hypotheses[13,12] are physical notions which promise to "explain" very successfully the structure of the turbulence (at high Reynolds number) at any instant. When we obtain equally convincing physical ideas about the way in which the governing parameters of this structure change from instant to instant, then our progress in the problem of isotropic turbulence will be considerable.

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The author wishes to thank Professor Sir Geoffrey Taylor for his interest in the work.

THE GENERAL PROBLEM OF ANTENNA RADIATION AND THE FUNDAMENTAL INTEGRAL EQUATION, WITH APPLICATION TO AN ANTENNA OF REVOLUTION—PART I*

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Introduction. The main result of Part I is the integral equation (3.17)—a general and exact equation satisfied by the current flowing in a perfectly conducting radiating antenna, the surface of which is a surface of revolution, but otherwise unrestricted except for implicit conditions of smoothness. Part II deals with the solution of that integral equation.

No use is made of vector or scalar potentials; an approach through the electromagnetic vectors appears simpler and more direct, since a radiating field is determined by the values of the tangential component of the electric vector on the inner boundary of the infinite region.[1]

Section 1 gives the basic equations and boundary conditions, together with some formulae connected with a dipole. Section 2 deals with the case of radiation in a finite cavity; this simpler case points the way to the theory of antenna radiation. The basic integral equation (2.6) for cavity eigen-modes is obtained, as well as the integral equation (2.4) which has the same form as the basic equation for the exterior (antenna) case. Section 3 develops the general integral equation (3.1) or (3.2) for antenna radiation, which reduces to the comparatively simple explicit form (3.17) for an antenna of revolution.[2] In Sec. 4 some actual radiating systems are discussed from the standpoint that the tangential component of the electric vector determines the field.

Section 5 is devoted to a discussion of the gap. The gap is in a sense the most important feature of an antenna because it is the source of radiant energy. (No energy passes out through the surface of a perfect conductor.) Strangely enough, the gap has received scant attention in radiation theory. The recent work of Infeld[3] shows that the question of the gap is a most delicate one; a very narrow gap leads to zero impedance unless there is a compensating thinness of the antenna at the gap. Zero impedance is avoided in the present paper by keeping the gap long compared with the thickness of the antenna at the gap as in (5.7).

In Sec. 6 the unknown is changed from the current $I(z)$ to the relative current $\phi(z)$,

*Received March 31, 1947. This paper is based on a report written by one of the authors (J.L.S.), when at the University of Toronto. This report was issued in 1942 by the National Research Council of Canada. The method now used in Part I differs considerably from that used in the report.

[1]The importance of the tangential components of the electromagnetic vectors appears to have been first pointed out by H. M. MacDonald, *Electric waves*, Cambridge, 1902, p. 16.

[2]For other, and we believe less exact integral equations in the antenna problem, see E. Hallén, Nova Acta Reg. Soc. Sci. Upsaliensis, **11**, No. 4 (1939); P. Nicolas, L'Onde électrique, **18**, 193-211 (1939); L. Brillouin, Quarterly Appl. Math. **1**, 201-214 (1943).

[3]L. Infeld, Quarterly Appl. Math. **5**, 113-132 (1947).

viz. the ratio of the current at z to the current at the center of the gap. Formulae (6.5), (6.14) give the impedance Z in terms of $\phi(z)$. The function $\phi(z)$ satisfies the integral equation (6.3), not essentially different from (3.17).

The thin antenna is discussed in Sec. 7. We are able to show that the current is approximately sinusoidal outside the gap (7.10). The sinusoidal distribution of current is commonly accepted, but proofs up to the present have not been satisfactory. The present method is valid for *any* thin antenna of revolution; it is not assumed to be cylindrical, except at the gap.

The calculations in the case of a finite gap are too complicated to carry out at present, and for simplicity the gap is assumed to be short in Sec. 8. Whatever the practical validity of the final results may be, it must be clearly understood that for theoretical validity we must have

$$\text{wave length} \gg \text{length of gap} \gg \text{radius of antenna}.$$

Equation (8.10) gives in compact form the principal part of the impedance of any thin antenna with such a gap anywhere in it. If the gap is taken at the center and the shape term X_s neglected—it is negligible for a cylindrical antenna with plane ends—we get the formula obtained by the less rational method of Labus.[4]

In Sec. 9 the shape term is discussed, with reference to the following thin antennae: (i) a cylinder with spheroidal ends, (ii) a spheroid, (iii) a cone. It is remarkable that the half-wave spheroid has zero reactance.

Section 10 contains a method of successive approximations for dealing with any antenna, thin or thick. The first approximations for the current and impedance agree with the principal parts obtained in Sec. 7 for the thin antenna. Higher approximations have not been calculated.

The principal features of the paper may be summarized as follows:

1) Radiation problems are reduced to the solution of integral equations involving the tangential components of the electric and magnetic vectors.
2) The gap is treated more adequately than has been done before.
3) The approximately sinusoidal current distribution for a thin antenna is established. The usually troublesome question of end effects disappears automatically in the present method.
4) The usual first approximation for the impedance of a thin antenna is obtained with the following generalizations:
   (a) The antenna is not necessarily cylindrical with plane ends.
   (b) The gap is not necessarily at the center.
5) A method of successive approximations is given for dealing with thick antennae.

Advice and criticism from the following is gratefully acknowledged: Professors L. Infeld, V. G. Smith, and A. F. Stevenson, and Dr. W. Z. Chien, at the University of Toronto; and Professors A. E. Heins, and A. Weinstein, at the Carnegie Institute of Technology. Computations for the graphs were made by Mr. George Succop.

1. Notation and preliminary equations. We confine our attention to simple harmonic electromagnetic fields in vacuo; the electric and magnetic vectors are the real parts of

$$\mathbf{E} \exp(-ikct), \qquad \mathbf{H} \exp(-ikct), \tag{1.1}$$

[4] J. Labus, Hochfrequenztechnik und Elektroakustik, 41, 17-23 (1933); G. H. Brown and R. King, Proc. I. R. E., 22, 457-480 (1934).

where $\mathbf{E}$ and $\mathbf{H}$ are vector functions of position, in general complex, c is the velocity of light, and k is a real constant related to the wave length λ by $\lambda k = 2\pi$. Except for the statement of some final results, we shall use Heaviside units, so that Maxwell's equations read

$$-ik\mathbf{E} = \nabla \times \mathbf{H}, \qquad ik\mathbf{H} = \nabla \times \mathbf{E}, \tag{1.2}$$

$$\nabla \cdot \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{H} = 0. \tag{1.3}$$

Equations (1.3) are necessarily satisfied if (1.2) are satisfied.

As regards boundary conditions, we have on the surface of a perfect conductor, with unit normal $\mathbf{n}$,

$$\mathbf{n} \times \mathbf{E} = 0, \tag{1.4}$$

i.e. $\mathbf{E}_t = 0$, where $\mathbf{E}_t$ is the tangential vector component of $\mathbf{E}$. It follows from the second of (1.2) that on the surface of a perfect conductor

$$\mathbf{H} \cdot \mathbf{n} = 0. \tag{1.5}$$

In dealing with an exterior problem, conditions at infinity must be stated. Instead of the usual Sommerfeld conditions, we shall take the following as conditions for outward radiation at infinity. *A constant B exists such that at all sufficiently great distances ρ from some fixed point O we have*

$$|\mathbf{E}| < B/\rho, \qquad |\mathbf{H}| < B/\rho, \tag{1.6}$$

$$|\mathbf{E} + \mathbf{n} \times \mathbf{H}| < B/\rho^2, \qquad |\mathbf{H} - \mathbf{n} \times \mathbf{E}| < B/\rho^2, \tag{1.7}$$

where $\mathbf{n}$ is the unit vector drawn from O towards the point at which the field is considered.

It is clear that these conditions are invariant under bounded displacement of the point O; this causes a bounded change in ρ and a change in $\mathbf{n}$ of order $1/\rho$.

The following extended Poynting's theorem, due to H. A. Lorentz,[5] is basic in our work: *Let $(\mathbf{E}, \mathbf{H})$ and $(\mathbf{E}', \mathbf{H}')$ be two Maxwellian fields in a finite region R bounded by a surface S, and let neither field have any singularities in R or on S. Then*

$$\int_S \mathbf{n} \cdot (\mathbf{E} \times \mathbf{H}' - \mathbf{E}' \times \mathbf{H})\, dS = 0, \tag{1.8}$$

where $\mathbf{n}$ is the unit normal vector to S. This is proved very easily by Green's theorem and Maxwell's equations (1.2).

Consider now the case where the region R extends to infinity, and is bounded internally by one or more surfaces S. Then if Σ is any large sphere of radius ρ enclosing S, we have

$$\int_S \mathbf{n} \cdot (\mathbf{E} \times \mathbf{H}' - \mathbf{E}' \times \mathbf{H})\, dS = \int_\Sigma \mathbf{n} \cdot (\mathbf{E} \times \mathbf{H}' - \mathbf{E}' \times \mathbf{H})\, dS, \tag{1.9}$$

where $\mathbf{n}$ denotes the outward unit normal vectors to S and to Σ. Suppose that both fields satisfy the conditions (1.6), (1.7) for outward radiation at infinity. Then, denoting by T_3 terms of order $1/\rho^3$, we have on Σ

[5]Cf. Frank and Mises, Differential- und Integralgleichungen der Mechanik und Physik, Vol. 2, Braunschweig, 1927, p. 576.

$$\begin{aligned}\mathbf{n}\cdot(\mathbf{E}\times\mathbf{H}'-\mathbf{E}'\times\mathbf{H}) &= -\mathbf{n}\cdot[(\mathbf{n}\times\mathbf{H})\times\mathbf{H}'-(\mathbf{n}\times\mathbf{H}')\times\mathbf{H}]+\mathbf{T}_3\\ &= -\mathbf{n}\cdot[\mathbf{H}(\mathbf{n}\cdot\mathbf{H}')-\mathbf{n}(\mathbf{H}\cdot\mathbf{H}')\\ &\qquad -\mathbf{H}'(\mathbf{n}\cdot\mathbf{H})+\mathbf{n}(\mathbf{H}'\cdot\mathbf{H})]+\mathbf{T}_3\\ &= \mathbf{T}_3\,.\end{aligned} \tag{1.10}$$

Hence, as ρ tends to infinity, the integral over Σ tends to zero, and we have this result: *Let* $(\mathbf{E}, \mathbf{H})$ *and* $(\mathbf{E}', \mathbf{H}')$ *be two Maxwellian fields in an infinite region R bounded internally by one or more surfaces S; let neither field have any singularities in R or on S, and let each field satisfy the conditions* (1.6), (1.7) *for outward radiation at infinity; then*

$$\int_S \mathbf{n}\cdot(\mathbf{E}\times\mathbf{H}'-\mathbf{E}'\times\mathbf{H})\,dS = 0. \tag{1.11}$$

Suppose now that there is a dipole of vector strength $\mathbf{A}$ at the point P'; the field due to it at a point P is

$$\mathbf{E}' = \nabla(\mathbf{A}\cdot\nabla\psi)+k^2\psi\mathbf{A}, \qquad \mathbf{H}' = ik\mathbf{A}\times\nabla\psi, \qquad \psi = e^{ikPP'}/PP'. \tag{1.12}$$

This field plays an important part in our work. Its essential features are: (i) it satisfies Maxwell's equations, (ii) it has a pole of a certain type at P', (iii) it satisfies the conditions (1.6), (1.7) at infinity. Let $(\mathbf{E}, \mathbf{H})$ be any Maxwellian field, regular in the neighborhood of P', and let σ be a spherical surface with center at P' and small radius ρ. Then

$$\lim_{\rho\to 0}\int_\sigma \mathbf{n}\cdot(\mathbf{E}\times\mathbf{H}'-\mathbf{E}'\times\mathbf{H})\,dS = 4\pi ik\mathbf{E}\cdot\mathbf{A}, \tag{1.13}$$

where $\mathbf{E}$ on the right hand side is evaluated at P', and $\mathbf{n}$ is the unit outward normal to σ. This limit was incorrectly calculated by Sommerfeld;[6] the error has been pointed out by A. F. Stevenson in a paper in course of publication,* and the correct limit (1.13) given.

2. The integral equation for cavity radiation. Interior (cavity) and exterior (antenna) problems in electromagnetic radiation present some common features and some striking

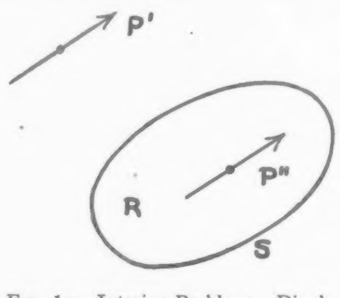

FIG. 1a. Interior Problem: Dipole at P' yields integral equation. Dipole at P'' yields field at P''.

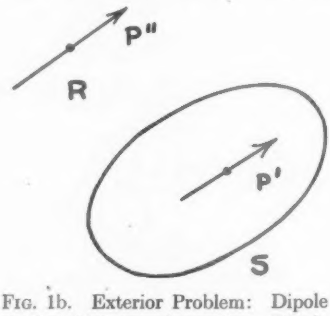

FIG. 1b. Exterior Problem: Dipole at P' yields integral equation. Dipole at P'' yields field at P''.

[6]P. Frank and R. v. Mises, op. cit., 578-580.

*Footnote added in proof: A. F. Stevenson, Q. Appl. Math. **5**, 369-384 (1948), in particular p. 377.

differences. Although the problem with which we are primarily concerned is the problem of antenna radiation, we shall first discuss cavity radiation.

Let R be a finite region (cavity) bounded by a surface S (Fig. 1a). Let $(\mathbf{E}, \mathbf{H})$ be a Maxwellian field inside R, satisfying (1.2); so far we do not mention boundary conditions. Let $(\mathbf{E}', \mathbf{H}')$ be the field due to a dipole situated at P', *outside* R. Then, as in (1.8),

$$\int_S \mathbf{n}\cdot(\mathbf{E} \times \mathbf{H}' - \mathbf{E}' \times \mathbf{H})\, dS = 0. \tag{2.1}$$

Let us write

$$\tilde{\mathbf{E}}_t = \mathbf{n} \times \mathbf{E}, \qquad \tilde{\mathbf{H}}_t = \mathbf{n} \times \mathbf{H}, \tag{2.2}$$

so that $\tilde{\mathbf{E}}_t$, $\tilde{\mathbf{H}}_t$ are vectors tangent to S; they are respectively the tangential vector components $\mathbf{E}_t$, $\mathbf{H}_t$ of $\mathbf{E}$, $\mathbf{H}$, turned through a right angle about $\mathbf{n}$ in the positive sense. Then (2.1) may be written

$$\int_S (\tilde{\mathbf{E}}_t\cdot\mathbf{H}' + \tilde{\mathbf{H}}_t\cdot\mathbf{E}')\, dS = 0, \tag{2.3}$$

or

$$\int_S (\tilde{\mathbf{E}}_t\cdot\mathbf{H}_t' + \tilde{\mathbf{H}}_t\cdot\mathbf{E}_t')\, dS = 0, \tag{2.4}$$

where $\mathbf{E}_t'$, $\mathbf{H}_t'$ are the tangential vector components of the dipole field.

Noting that the vector strength and the position of the dipole are arbitrary (subject only to the condition that it must lie outside R), we see that (2.4) *is an integral equation satisfied by the tangential components on S of any Maxwellian field regular inside S.*

Once $(\tilde{\mathbf{E}}_t, \tilde{\mathbf{H}}_t)$ are known on S, we can find the field $(\mathbf{E}, \mathbf{H})$ at any point P'' in the cavity as follows. We set up a dipole of strength $\mathbf{A}$ at P''; let $(\mathbf{E}'', \mathbf{H}'')$ be the field of this dipole. Then we apply (2.1) to the region bounded outside by S and inside by a small sphere centered at P''. Then, if $\mathbf{n}$ is the unit normal to S, drawn into the cavity, we have by (1.13)

$$4\pi ik(\mathbf{E})_{P''}\cdot\mathbf{A} = -\int_S (\tilde{\mathbf{E}}_t\cdot\mathbf{H}_t'' + \tilde{\mathbf{H}}_t\cdot\mathbf{E}_t'')\, dS. \tag{2.5}$$

The integral equation (2.4) may be used to study forced oscillations of a cavity, or free oscillations. Suppose, for example, that the cavity is excited by given $\mathbf{E}_t$ on its surface; then (2.4) becomes an integral equation to determine $\mathbf{H}_t$, and, if it can be solved, the field in the cavity is given by (2.5). In the case of free oscillations in a cavity with perfectly conducting walls, we put $\mathbf{E}_t = 0$; then (2.4) reads

$$\int_S \tilde{\mathbf{H}}_t\cdot\mathbf{E}_t'\, dS = 0. \tag{2.6}$$

It is well known that free oscillations in a cavity are possible only for definite values (eigenvalues) of k; (2.6) is an integral equation of an unusual type for the determination of such eigenvalues. But we shall not discuss the question of eigenvalues here.[7]

[7]Cf. H. Weyl, Journal für Mathematik, **143**, 177-202 (1914); Rendiconti del Circolo Matematico id Palermo, 39, 1-50 (1915).

The reader should remember that, in the above work,

$$(\mathbf{E}', \mathbf{H}') = \text{field due to dipole outside the cavity;}$$

$$(\mathbf{E}'', \mathbf{H}'') = \text{field due to dipole placed in the cavity.}$$

3. The integral equation for antenna radiation. Let R be an infinite region bounded internally by a surface S. We are interested in simply harmonic Maxwellian fields, satisfying the conditions (1.6), (1.7) of outward radiation at infinity and certain conditions over the surface S (Fig. 1b). In the physical problem, S is a surface enclosing the antenna or coincident with it.

Let $(\mathbf{E}, \mathbf{H})$ be the field under discussion. Let $(\mathbf{E}', \mathbf{H}')$ be the field of a dipole of vector strength $\mathbf{A}$ situated at any point P' *inside* S, i.e. not in the region R. We may apply the result (1.8) to the surface composed in part of S and in part of a large sphere. As we have already seen, the contribution from the large sphere tends to zero as its radius tends to infinity. Hence, in the notation of (2.4) we have

$$\int_S (\tilde{\mathbf{E}}_t \cdot \mathbf{H}'_t + \tilde{\mathbf{H}}_t \cdot \mathbf{E}'_t)\, dS = 0, \tag{3.1}$$

where t signifies "tangential vector component" and $\tilde{\mathbf{E}}_t$, $\tilde{\mathbf{H}}_t$ denote that $\mathbf{E}_t$, $\mathbf{H}_t$ are turned through a right angle in the positive sense about the unit normal $\mathbf{n}$ to S. We shall in future regard $\mathbf{n}$ as drawn out from S, i.e. into R. It is clear that $\tilde{\mathbf{E}}_t \cdot \mathbf{H}'_t = -\mathbf{E}_t \cdot \tilde{\mathbf{H}}'_t$, and so (3.1) may be written

$$\int_S \tilde{\mathbf{H}}_t \cdot \mathbf{E}'_t\, dS = \int_S \mathbf{E}_t \cdot \tilde{\mathbf{H}}'_t\, dS. \tag{3.2}$$

Equation (3.1) or (3.2) is the *fundamental integral equation of antenna theory.* If this equation is solved, and $\mathbf{E}_t$, $\mathbf{H}_t$ found, the field at any point in R is obtained by taking a dipole of vector strength $\mathbf{A}$ at any point P'' inside R, i.e. outside S. Then, as in (2.5), we have

$$4\pi i k(\mathbf{E})_{P''} \cdot \mathbf{A} = -\int_S (\tilde{\mathbf{E}}_t \cdot \mathbf{H}''_t + \tilde{\mathbf{H}}_t \cdot \mathbf{E}''_t)\, dS, \tag{3.3}$$

when $(\mathbf{E}'', \mathbf{H}'')$ is the field due to the dipole placed at P''; the normal on S points outward.

Let us now apply our equation (3.2) to the case of an antenna having symmetry of revolution about the z-axis (Fig. 2).

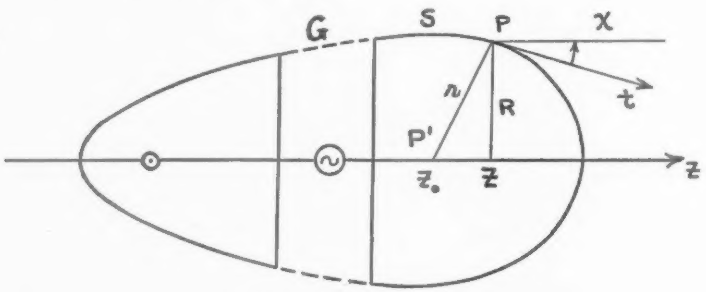

Fig. 2. Antenna of revolution with gap.

The antenna is broken by a gap containing a generator. Let the surface of the antenna be extended by a mathematical surface G covering the gap, and let S be a closed surface consisting in part of the surface of the antenna and in part of the surface G drawn over the gap. We shall suppose the excitation of the antenna to be symmetric in the sense that, on G, $\mathbf{E}_t$ lies in the meridian plane. It is natural, then, to consider only fields of radiation having this type of symmetry, i.e. $\mathbf{E}$ at any point P lies in the meridian plane through P and the z-axis, and $\mathbf{H}$ in consequence is perpendicular to the meridian plane. On S, $\tilde{\mathbf{H}}_t$ lies in the meridian plane.

Now take a dipole at any point $P'(z_0)$ on the axis of the antenna inside S, the axis of the dipole lying along the axis of the antenna, and its strength being unity. We shall apply the fundamental equation (3.2).

Introducing cylindrical coordinates (z, R, ω), we have, by (1.12),

$$\psi = e^{ikr}/r, \qquad r^2 = R^2 + (z - z_0)^2. \tag{3.4}$$

The surviving components of the field of the dipole are

$$E'_z = \frac{\partial^2 \psi}{\partial z^2} + k^2\psi, \qquad E'_R = \frac{\partial^2 \psi}{\partial R\, \partial z}, \qquad H'_\omega = ik\frac{\partial \psi}{\partial R}, \tag{3.5}$$

where the differentiations of ψ are carried out with R, z, z_0 treated as independent variables. However, on the surface of the antenna, R is a function of z, and so we may write $\psi = \psi(z, z_0)$. Let us temporarily employ d as a symbol of partial differentiation when ψ and its derivatives are regarded as functions of z and z_0. Then, noting that

$$\frac{d}{dz_0}\frac{\partial \psi}{\partial R} = \frac{\partial}{\partial z_0}\frac{\partial \psi}{\partial R} = -\frac{\partial}{\partial z}\frac{\partial \psi}{\partial R},$$

$$\frac{d}{dz_0}\frac{\partial \psi}{\partial z} = \frac{\partial}{\partial z_0}\frac{\partial \psi}{\partial z} = -\frac{\partial}{\partial z}\frac{\partial \psi}{\partial z},$$

we have

$$\begin{aligned}\frac{d\psi}{dz} &= \frac{\partial \psi}{\partial R}\frac{dR}{dz} + \frac{\partial \psi}{\partial z},\\ \frac{d^2\psi}{dz_0\, dz} &= -\frac{\partial^2 \psi}{\partial R\, \partial z}\frac{dR}{dz} - \frac{\partial^2\psi}{\partial z^2}.\end{aligned} \tag{3.6}$$

The tangential component $\mathbf{E}'_t$ may be written $\mathbf{E}'_t = E'\mathbf{t}$, where $\mathbf{t}$ is a unit tangent vector to the meridian curve, as shown in Fig. 2; $\mathbf{t}$ makes an angle χ with the z-axis, so that

$$\sin\chi = -\frac{dR/dz}{[1 + (dR/dz)^2]^{1/2}}, \qquad \cos\chi = \frac{1}{[1 + (dR/dz)^2]^{1/2}}. \tag{3.7}$$

Then, by (3.5) and (3.6),

$$\begin{aligned}E' &= E'_z \cos\chi - E'_R \sin\chi\\ &= [1 + (dR/dz)^2]^{-1/2}\left[\frac{\partial^2\psi}{\partial z^2} + k^2\psi + \frac{\partial^2\psi}{\partial R\, \partial z}\frac{dR}{dz}\right]\\ &= [1 + (dR/dz)^2]^{-1/2}\left[k^2\psi - \frac{d^2\psi}{dz\, dz_0}\right].\end{aligned} \tag{3.8}$$

If we write $\tilde{\mathbf{H}}_t = \tilde{H}\mathbf{t}$, we have

$$\tilde{\mathbf{H}}_t \cdot \mathbf{E}_t' = E'\tilde{H} = [1 + (dR/dz)^2]^{-1/2}\left[k^2\psi - \frac{d^2\psi}{dz\,dz_0}\right]\tilde{H}. \tag{3.9}$$

Also, we may write $\mathbf{E}_t = E\mathbf{t}$, $\tilde{\mathbf{H}}_t' = \tilde{H}'\mathbf{t}$, where, by (3.5),

$$\tilde{H}' = H_\omega' = ik\frac{\partial\psi}{\partial R}. \tag{3.10}$$

Now ψ is a function of r, a single variable, and $d\psi/dr$ will denote the derivative. We have

$$\frac{\partial\psi}{\partial R} = \frac{d\psi}{dr}\frac{\partial r}{\partial R} = \frac{R}{r}\frac{d\psi}{dr}, \tag{3.11}$$

and so

$$\mathbf{E}_t \cdot \tilde{\mathbf{H}}_t' = E\tilde{H}' = ikE\frac{R}{r}\frac{d\psi}{dr}. \tag{3.12}$$

Thus our basic equation (3.2) reads

$$\int_S \left[k^2\psi - \frac{d^2\psi}{dz\,dz_0}\right][1 + (dR/dz)^2]^{-1/2}\tilde{H}\,dS = ik\int_S \frac{R}{r}\frac{d\psi}{dr}E\,dS. \tag{3.13}$$

Let us denote by $I(z)$ the current flowing on the surface of the antenna across the section z = constant. Then

$$I(z) = 2\pi cR\tilde{H}. \tag{3.14}$$

We shall regard this as the definition of $I(z)$ on G. Also

$$dS = 2\pi R[1 + (dR/dz)^2]^{1/2}\,dz. \tag{3.15}$$

Hence (3.13) may be written

$$\int \left(k^2\psi - \frac{d^2\psi}{dz\,dz_0}\right)I(z)\,dz = 2\pi ikc\int R^2[1 + (dR/dz)^2]^{1/2}\frac{1}{r}\frac{d\psi}{dr}E(z)\,dz. \tag{3.16}$$

This equation (3.16) is the basic integral equation for the determination of the current in an antenna of revolution. We shall therefore restate it in complete form with slightly revised notation: *Consider a perfectly conducting antenna having the z-axis for axis of symmetry, and excited symmetrically. Let the antenna extend from* $z = l_1$ *to* $z = l_2$. *Let* $R = R(z)$ *be the equation in cylindrical coordinates of a surface S which consists of the surface of the antenna and the gap G. Let* $E(z)$ *be the tangential component of the electric vector on the gap, and* $E(z) = 0$ *outside the gap. Let* $I(z)$ *be the current across the section* z = *const. as given by* (3.14). *Then* $I(z)$ *satisfies the integral equation*

$$\int_{l_1}^{l_2} K(z, z_0)I(z)\,dz = iM(z_0), \qquad l_1 < z_0 < l_2, \tag{3.17}$$

where

$$K(z, z_0) = -\frac{\partial^2\psi}{\partial z\,\partial z_0} + k^2\psi,$$

$$\psi = \psi(z, z_0) = e^{ikr}/r, \qquad r^2 = [R(z)]^2 + (z - z_0)^2, \tag{3.18}$$

$$M(z_0) = 2\pi kc \int_{l_1}^{l_2} ER^2(1 + R'^2)^{1/2} \frac{1}{r} \frac{d\psi}{dr} dz,$$

$$R' = dR/dz.$$

Since confusion with regard to partial differentiation has now been removed, we have restored the usual notation, ψ being regarded as a function of z and z_0. It should be noted that on account of the occurrence of $R(z)$ in r, the kernel $K(z, z_0)$ is not symmetric. Since $E = 0$ outside the gap, the limits of integration in the expression for $M(z_0)$ may be replaced by those corresponding to the ends of the gap.

4. Some radiation problems. In setting up simplified mathematical models for the treatment of radiation problems, it is essential that we carry into the model sufficient conditions to yield a definite mathematical problem, without putting in so many conditions as to make the problem insoluble. The following discussion is based on the fact that a field of outward radiation is determined by the assignment of $\mathbf{E}_t$ over a surface S which forms the inner boundary of the infinite region.

The mast antenna. The common problem of macrowave radiation involves a vertical mast above a horizontal earth (Fig. 3). Let us make the usual assumption of infinite conductivity. We think, then, of an infinitely conducting cylinder AB above an infinitely conducting plane. The apparatus is fed by a generator D, the terminals of which

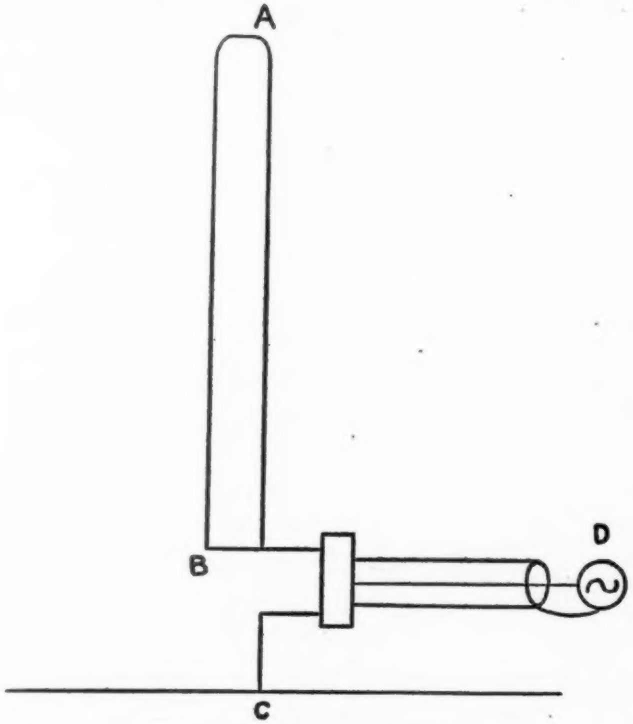

FIG. 3. The mast antenna

are connected to AB and to the plane at C. We have then the problem of determining a Maxwellian field satisfying the following conditions:

(i) outward radiation at infinity,

(ii) $\mathbf{E}_t = 0$ over AB, over the plane earth, and over the connecting leads.

Where does the energy come from, which reaches infinity? It cannot come from the perfectly conducting surfaces, since $\mathbf{E}_t = 0$ implies that the Poynting vector is tangent to a perfectly conducting surface. Hence the only possible source of the energy is a surface drawn around the generator D. (If that surface were perfectly conducting, the generator would be short-circuited, and there would be no radiation at all.) The rest of the apparatus serves only as a guide to the radiant energy.

It would appear therefore that we can obtain a determinate problem in the case of Fig. 3 only by assigning $\mathbf{E}_t$ over a surface which bounds the generator. If this were done, we would have $\mathbf{E}_t$ all over a surface consisting partly of the aforesaid surface and partly of the surfaces of the conductors, and so we would have a determinate problem.

But in the actual treatments of such problems, no reference is made to the geometry of a surface bounding the generator, nor even (usually) to the geometry of the leads. A discussion along the lines suggested above would be, apparently, unnecessarily complicated from a practical viewpoint. It is usual to simplify the problem by substituting a model as shown in Fig. 4 for the actual system. Now the generator D is reduced to

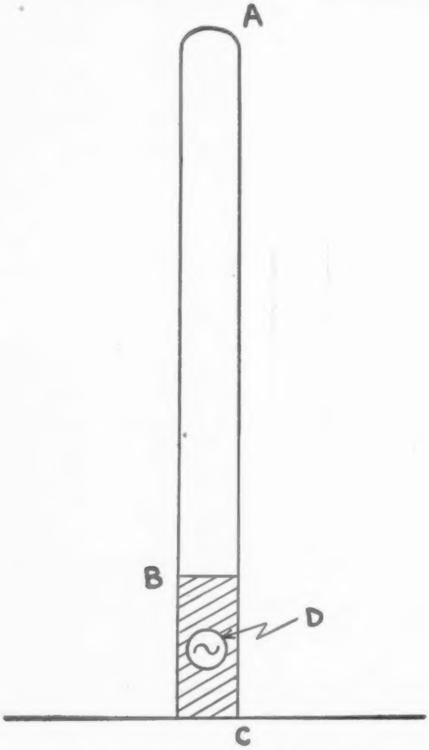

FIG. 4. The mast antenna with gap.

small proportions and placed between the antenna and the ground. To provide a surface for boundary conditions, the geometrical surface of the antenna is continued down past the generator to the ground.

It is not, of course, suggested that the above arrangement is ever employed in practice. But it is the arrangement employed in mathematical discussions, whether the authors admit it or not. The hope is that the behavior of the model shown in Fig. 4 (deduced mathematically) will agree with the behavior of the actual apparatus of Fig. 3. On this point we have no assurance from theory.

Figure 4 introduces the *gap*; by this we mean the surface BC, bounding the generator. Many writers have proceeded as if antenna problems could be discussed without reference to a gap. To do so, however, is to leave out the essential part of a radiating system. *The gap is the only source of radiant energy.*

Now if we knew $\mathbf{E}_t$ over the gap, we would have a determinate problem in Fig. 4. But we do not know $\mathbf{E}_t$ over the gap. This may be expressed by saying that we do not know the (electrical) *structure of the gap.* But if the gap is short, its structure does not seem to be important. It is then possible to disregard the structure, and express the radiation properties of the antenna in terms of only one characteristic of the gap, namely, the *voltage* or *potential difference.* This is the integral of $\mathbf{E}_t$ taken along BC, with suitable sign.

The apparatus used by Brown and Woodward[8] for the experimental measurement of impedance is similar to that shown in Fig. 4, except that their gap is shorter and the mast is fed by a coaxial line which comes in from below through the ground.

The cigar antenna. It is customary and legitimate to eliminate the earthplane in Fig. 4 by a method of symmetry. This leads to the cigar antenna (Fig. 5). This self-

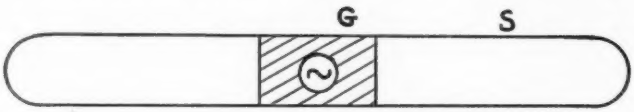

FIG. 5. The cigar antenna.

contained radiator is a mathematical fiction, but it could be realized in practice. We think of a hollow copper cigar, cut in two to leave a gap. Inside there is a generator, with terminals connected to the two ends of the cigar. (The two generators obtained by reflection of Fig. 4 in the earth are replaced by one generator in Fig. 5; but this is a triviality, because we are not interested in the interior of the cigar. What interests us is $\mathbf{E}_t$ over the gap G.)

Although the source of the cigar antenna is to be found in the application of the reflection process to the mast antenna, it suggests a generalization in which symmetry is abandoned (Fig. 6). Figure 7 shows a coaxial line terminated in a spherical knob. The model for this (Fig. 8) is the limiting case of an asymmetric cigar, one end being reduced to a small circular patch on a conducting sphere, surrounded by a ring gap.

The later detailed applications in this paper deal with a cigar antenna, in general asymmetric.

The microwave antenna terminating a coaxial line. Let us now consider a microwave

[8] G. H. Brown and O. M. Woodward, Proc. I. R. E., **33**, 257-262 (1945).

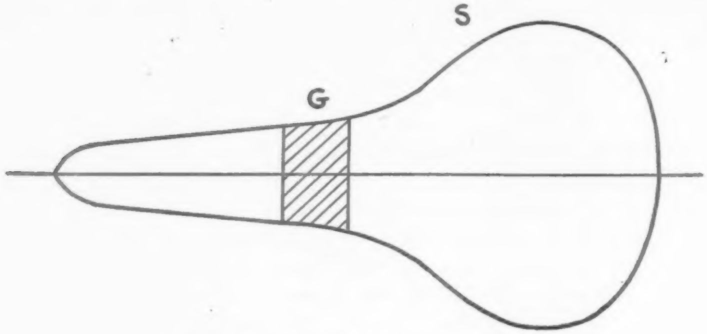

FIG. 6. The cigar antenna (unsymmetric).

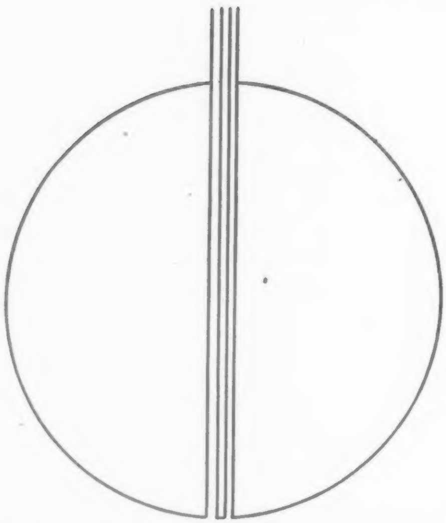

FIG. 7. Coaxial line terminated by a spherical knob.

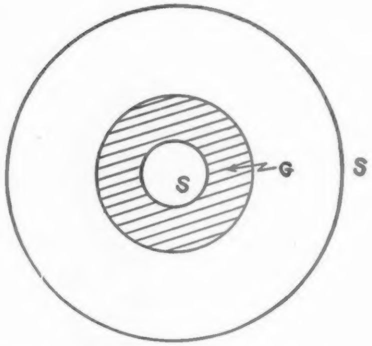

FIG. 8. Spherical antenna with ring gap.

apparatus. Figure 9 shows a coaxial line, terminated in an antenna. We may introduce the idealization that the line extends to infinity.

Let us see whether this apparatus gives a definite problem. For surface S we take the surface of the antenna and the core of the coaxial, together with the inner and outer surfaces of the sheath of the coaxial. Over this surface we have $\mathbf{E}_t = 0$. Conditions at infinity, however, are not obvious. We cannot say that there is outward radiation in all directions—there is nowhere for this radiation to come from. We expect that there will be outward radiation everywhere except in the neighbourhood of the coaxial line, and that inside the line there will be *inward* radiation (together with some outward radiation if there is incomplete tuning).

Thus, while Fig. 9 presents difficulties as far as the definition of a determinate mathematical problem is concerned, these difficulties are not those encountered previously. There is no troublesome gap—unless we like to say that the gap is at infinity.

Obviously some simplification is necessary for mathematical attack. The usual simplification is drastic. We retain only the horizontal arms of the antenna, wiping out all the rest of the apparatus. We join the arms (dotted indication in Fig. 9) and regard this cylindrical boundary as a gap. In fact, we return to the cigar antenna! It is indeed remarkable that impedance calculations obtained from such a simplified model should have the practical value that they do seem to possess.

The idealization of the microwave antenna of Fig. 9 to the cigar antenna of Fig. 5 involves us in the old difficulties concerning the gap. This seems particularly objectionable because in actual antennae the gap is not very small compared to the length of the antenna. Is there not some way to discuss the problem without introducing a gap artificially? It appears difficult in the case of the arrangement of Fig. 9.

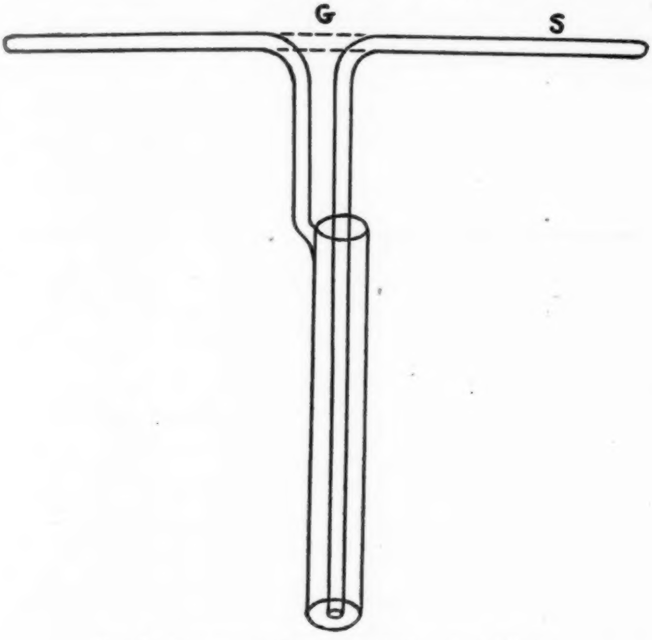

Fig. 9. Two-arm antenna terminating coaxial line.

With a different form of apparatus, however, the situation is more hopeful. Figure 10 shows a coaxial line, with the core projecting as an antenna, while the sheath spreads out into a conducting plane. The surface S consists of the surfaces of the core and antenna, with the inner surface of the sheath and the surface of the conducting plane. Over S we have $\mathbf{E}_t = 0$. On the infinite hemisphere above the antenna we assign outward radiation. At infinity down the line we assign a direct wave (amplitude C_1) and a reflected wave (amplitude C_2). The problem of determining the ratio C_1/C_2 is then a definite one. We approach the important problem of reflection without using the artificial concept of impedance.

To bring the preceding problem more into line with what was said earlier regarding determination, we may change the conditions down the line. Let us say that $\mathbf{E}_t$ is assigned

on a cross-section a long way down. Now we have a definite problem, since $\mathbf{E}_t$ is given over the bounding surface, with a condition of outward radiation in the hemisphere. If we find $\mathbf{H}_t$ on the section of the line, we can find the ratio of the reflected to the incident wave.

The case of Fig. 10 should therefore be treated with considerable respect, because it represents an actual situation in which the radiation problem is definite; it is unnecessary to simplify the apparatus drastically for mathematical purposes.[9]

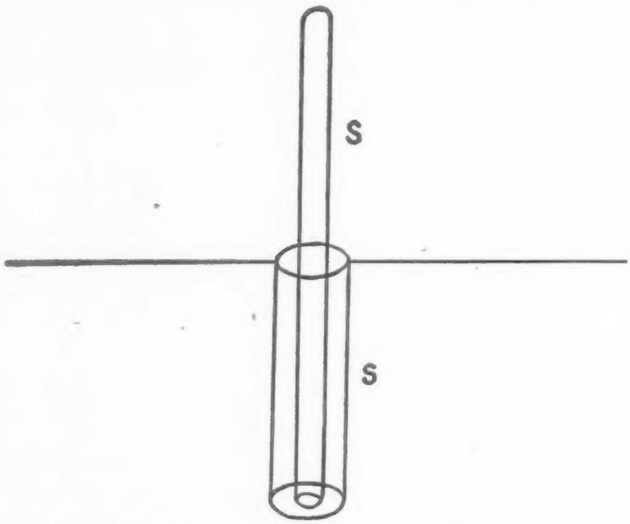

Fig. 10. Coaxial line with antenna projecting beyond conducting plate.

Before leaving the case of Fig. 10 we may mention how this apparatus may be considered as of the gap type. If we consider only phenomena above the plane of the flat conducting surface, we have a situation very like that of the conducting mast; the only difference is that the gap is now around the base of the mast instead of being below it. Once more we may appeal to the method of symmetry and discuss the problem as a cigar problem (Fig. 11). Now the surface S consists of a complete cigar with a disc attached at its center. We must assume $\mathbf{E}_t$ given over this disc, which is in fact the gap in a new form. We are not to expect that the field has continuous derivatives if we pass through the disc. We may call Fig. 5 a cigar-with-a-band and Fig. 11 a cigar-with-a-frill.

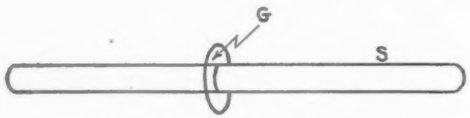

Fig. 11. Cigar with a frill.

[9]Experimental results for this case have been given by Brown and Woodward, *loc. cit.*

The antenna in the wave-guide. Figure 12 shows an antenna in a wave-guide. This problem lends itself to the same satisfactory mathematical formulation as for Fig. 10. There are only two differences. Instead of a hemisphere at infinity above, we have the opening of the wave guide at infinity on the right, and there we impose a condition of outward radiation. (That is in the case of transmission; for reception, we must allow for a direct and reflected wave, with only a wave *down* the coaxial line.) Secondly, the five faces of the guide (if it is of rectangular section) increase the complexity of the geometry.

Should we wish to introduce the gap concept, we may use the method of images

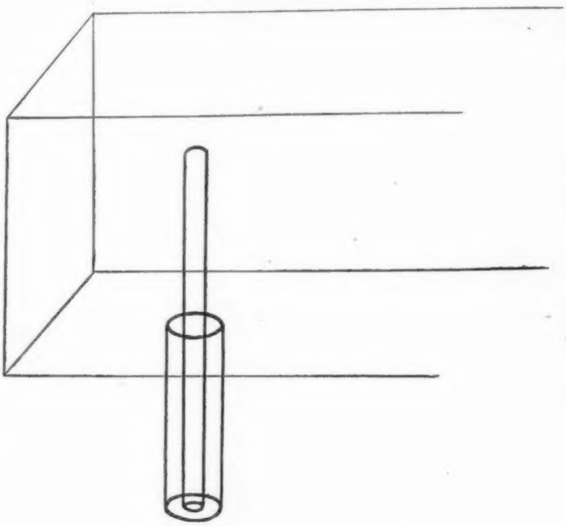

FIG. 12. Antenna in a wave-guide.

when the guide is rectangular. This means that we forget all about the walls of the guide and the coaxial line, and consider a doubly infinite pattern of cigars-with-frills in empty space.

These problems of radiation have been discussed in general terms at some length, because such discussions have unfortunately been lacking in the literature of radiation. It seems essential that we should have at least an idea of the conditions that make a problem definite from a mathematical viewpoint, and also that we should recognize the simplifications that are made to reduce practical problems to a form that admits mathematical treatment.

THE GENERAL PROBLEM OF ANTENNA RADIATION AND THE FUNDAMENTAL INTEGRAL EQUATION, WITH APPLICATION TO AN ANTENNA OF REVOLUTION—PART II*

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5. The gap. Let us assume the gap to be cylindrical, of radius $R = a$; let it extend from $z = -\eta$ to $z = \eta$. Let us suppose that on the gap

$$E = -\tfrac{1}{2}V/\eta, \qquad (V = \text{const.}) \tag{5.1}$$

so that V is the potential difference across the gap. Then, by (3.18),

$$M(z_0) = \pi c k^2 V N(z_0), \qquad N(z_0) = -\frac{a^2}{k\eta}\int_{-\eta}^{\eta} \frac{1}{r}\frac{d\psi}{dr}\,dz. \tag{5.2}$$

Now

$$\begin{gathered} r^2 = a^2 + (z - z_0)^2, \qquad \psi = r^{-1}e^{ikr}, \\ \frac{1}{r}\frac{d\psi}{dr} = -\frac{1}{r^3} - \frac{k^2}{2r} + k^3(\chi_1 + i\chi_2), \\ \chi_1 = \frac{1}{k^3r^3}(1 - \tfrac{1}{2}k^2r^2 - \cos kr) + \frac{1}{k^2r^2}(kr - \sin kr), \\ \chi_2 = \frac{1}{k^3r^3}(kr - \sin kr) - \frac{1}{k^2r^2}(1 - \cos kr). \end{gathered} \tag{5.3}$$

Note that χ_1, χ_2 are power series in positive powers of kr. We obtain at once

$$\begin{aligned} N(z_0) = {} & \frac{1}{k\eta}\left\{\frac{\eta - z_0}{[a^2 + (\eta - z_0)^2]^{1/2}} + \frac{\eta + z_0}{[a^2 + (\eta + z_0)^2]^{1/2}}\right\} \\ & + \frac{1}{2}\frac{k^2a^2}{k\eta}\{\ln k(\eta - z_0 + [a^2 + (\eta - z_0)^2]^{1/2}) \\ & + \ln k(\eta + z_0 + [a^2 + (\eta + z_0)^2]^{1/2}) - \ln k^2a^2\} \\ & - \frac{k^2a^2}{\eta}\int_{-\eta}^{\eta}(\chi_1 + i\chi_2)\,dz. \end{aligned} \tag{5.4}$$

We shall now make two important simplifying assumptions. The first is

$$ka \ll 1. \tag{5.5}$$

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This means that the radius of the antenna at the gap is small compared with the wave length. Then we have approximately

$$N(z_0) = \frac{1}{k\eta}\left\{\frac{\eta - z_0}{[a^2 + (\eta - z_0)^2]^{1/2}} + \frac{\eta + z_0}{[a^2 + (\eta + z_0)^2]^{1/2}}\right\}, \tag{5.6}$$

provided a/η is not large. The second assumption is

$$a/\eta \ll 1. \tag{5.7}$$

This means that the gap is long compared with the radius of the antenna at the gap. Then (5.6) gives approximately

$$N(z_0) = \frac{1}{k\eta}\left\{\frac{\eta - z_0}{|\eta - z_0|} + \frac{\eta + z_0}{|\eta + z_0|}\right\}, \tag{5.8}$$

and so

$$\begin{aligned} N(z_0) &= \frac{2}{k\eta} \qquad \text{for } |z_0| < \eta, \\ N(z_0) &= 0 \qquad \text{for } |z_0| > \eta. \end{aligned} \tag{5.9}$$

Substitution in (5.2) gives

$$\begin{aligned} M(z_0) &= 2\pi ckV/\eta \qquad \text{for } |z_0| < \eta \qquad \text{(in gap)} \\ M(z_0) &= 0 \qquad \text{for } |z_0| > \eta \qquad \text{(outside gap)} \end{aligned} \tag{5.10}$$

6. Impedance and relative current. In this section we introduce the impedance Z and the relative current $\phi(z)$, and show how Z is found when $\phi(z)$ is known. The argument is exact; we understand by N the exact expression (5.2) rather than the approximation (5.9).

We define the impedance of the antenna to be

$$Z = V/I(0) \tag{6.1}$$

It may seem unnatural to use $I(0)$ in defining impedance. The point $z = 0$ is at the center of the gap, and there is no current there. In fact, $I(0)$ means $2\pi caH(0)$, according to (3.14). It might seem better to use the currents at the ends of the gap. But it appears simpler to use $I(0)$ as basic; we can easily pass to the other definitions, if required. In the case of a very short gap (Sec. 8), these subtle distinctions disappear, for we have then

$$I(-\eta) = I(0) = I(\eta)$$

approximately.

Let us write

$$\phi(z) = I(z)/I(0); \tag{6.2}$$

this will be called the *relative current.*

On dividing (3.17) by $I(0)$, we get

$$\int_{l_1}^{l_2} K(z, z_0)\phi(z)\, dz = i\pi ck^2 Z\, N(z_0), \qquad (l_1 < z_0 < l_2) \tag{6.3}$$

With this equation we associate the boundary conditions

$$\phi(0) = 1, \phi(l_1) = \phi(l_2) = 0. \tag{6.4}$$

The equation (6.3) contains the unknown function $\phi(z)$ and the unknown constant Z. If we knew $\phi(z)$, we could calculate Z at once, giving any value to z_0. If we have only a rough idea of $\phi(z)$, it would be better not to take a definite value of z_0, but to introduce a weighting factor $f(z_0)$, and calculate Z from

$$Z = -\frac{i}{\pi c k^2} \frac{\int_{l_1}^{l_2} f(z_0)\, dz_0 \int_{l_1}^{l_2} K(z, z_0)\phi(z)\, dz}{\int_{l_1}^{l_2} N(z_0) f(z_0)\, dz_0}. \tag{6.5}$$

For the present let us leave the weighting function $f(z)$ arbitrary except for the assumptions that it is continuous, has a continuous derivative, and satisfies the end conditions

$$f(l_1) = f(l_2) = 0. \tag{6.6}$$

Let us write J for the numerator in (6.5) and understand the limits of integration (l_1, l_2). Then

$$\begin{aligned} J &= \iint K(z, z_0)\phi(z) f(z_0)\, dz\, dz_0 \\ &= -\iint \frac{\partial^2 \psi}{\partial z\, \partial z_0} \phi(z) f(z_0)\, dz\, dz_0 + k^2 \iint \psi(z, z_0)\phi(z) f(z_0)\, dz\, dz_0 , \end{aligned} \tag{6.7}$$

and on integration by parts

$$J = \iint \psi(z, z_0)\chi(z, z_0)\, dz\, dz_0 , \tag{6.8}$$

where

$$\chi(z, z_0) = -\phi'(z) f'(z_0) + k^2 \phi(z) f(z_0). \tag{6.9}$$

Now we may write (6.8) as follows:

$$\begin{aligned} J &= J_1 + J_2 , \\ J_1 &= \iint \frac{1}{r(z, z_0)} \chi(z, z)\, dz\, dz_0 , \\ J_2 &= \iint \frac{1}{r(z, z_0)} \{\chi(z, z_0) \exp(ikr(z, z_0)) - \chi(z, z)\}\, dz\, dz_0 , \end{aligned} \tag{6.10}$$

$$[r(z, z_0)]^2 = [R(z)]^2 + (z - z_0)^2.$$

For a thin antenna, r is small for $z = z_0$. However, the integrand in J_2 remains finite as $R \to 0$. This is the reason for splitting J as above, and forms the basis of later approximations. For the present the argument remains exact.

The integral J_1 gives

$$J_1 = J_{11} + J_{12} + J_{13} , \tag{6.11}$$

where

$$J_{11} = L \int_{l_1}^{l_2} \chi(z, z)\, dz, \qquad L = -\ln(k^2 a^2),$$

$$J_{12} = \int_{l_1}^{l_2} \chi(z, z) \ln \frac{a^2}{(R(z))^2}\, dz, \tag{6.12}$$

$$J_{13} = \int_{l_1}^{l_2} \chi(z, z) \ln \{k^2[l_2 - z + (R^2 + (l_2 - z)^2)^{1/2}][z - l_1 + (R^2 + (z - l_1)^2)^{1/2}]\}\, dz.$$

Note, for later approximation, that J_{12}, J_{13} remain finite for an infinitely thin antenna.

We have, by (6.6),

$$\begin{aligned} J_{11} &= L \int_{l_1}^{l_2} \{-\phi'(z)f'(z) + k^2\phi(z)f(z)\}\, dz \\ &= L \int_{l_1}^{l_2} \{f''(z) + k^2 f(z)\}\phi(z)\, dz. \end{aligned} \tag{6.13}$$

Now (6.5) may be written

$$Z = -\frac{i}{\pi c k^2} \frac{J_{11}(\phi, f) + J_{12}(\phi, f) + J_{13}(\phi, f) + J_2(\phi, f)}{\int_{l_1}^{l_2} N(z) f(z)\, dz}. \tag{6.14}$$

This notation puts in evidence the dependence of the J's on the two functions—the relative current ϕ and the weighting function f.

The function f is at our disposal. We see from (6.13) that J_{11} would vanish for an f sinusoidal in kz. However, unless the whole length of the antenna is a multiple of $\frac{1}{2}\lambda$, there exists no such function with continuous derivative, satisfying the end conditions (6.6). So we approach a sinusoidal f by a limiting process, in which (on attaining the limit) the continuity of the derivative is lost, but (6.14) remains true.

Let ϵ be any small positive number. We define a function $f_1(z, \epsilon)$ as follows:

$$l_1 \le z \le -\epsilon : K(\epsilon) f_1(z, \epsilon) = -\sin kl_2 \sin k(l_1 - z), \tag{6.15a}$$

$$\begin{aligned} -\epsilon \le z \le \epsilon \ : K(\epsilon) f_1(z, \epsilon) = {} & K(\epsilon) \cos kz + \tfrac{1}{2} \sin kz \sin k(l_1 + l_2) \\ & - \tfrac{1}{2} \operatorname{cosec} k\epsilon \sin k(l_2 - l_1)(1 - \cos kz), \end{aligned} \tag{6.15b}$$

$$\epsilon \le z \le l_2 \ : K(\epsilon) f_1(z, \epsilon) = -\sin kl_1 \sin k(l_2 - z), \tag{6.15c}$$

$$K(\epsilon) = -\sin kl_1 \sin kl_2 - \tfrac{1}{2} \tan \tfrac{1}{2}k\epsilon \cdot \sin k(l_2 - l_1). \tag{6.16}$$

To avoid complicating the argument, we assume

$$\sin kl_1 \ne 0, \qquad \sin kl_2 \ne 0. \tag{6.17}$$

This means that neither arm of the antenna, measured from the center of the gap, is a multiple of $\frac{1}{2}\lambda$. Such critical cases must be approached by a special limiting process.

We note that $f_1(z, \epsilon)$ is continuous, with continuous first derivative, and satisfies

$$|z| > \epsilon : f_1''(z, \epsilon) + k^2 f_1(z, \epsilon) = 0, \tag{6.18a, c}$$

$$|z| < \epsilon : f_1''(z, \epsilon) + k^2 f_1(z, \epsilon) = -\frac{k^2 \sin k(l_2 - l_1)}{2K(\epsilon) \sin k\epsilon}. \tag{6.18b}$$

Then, by (6.13), in an obvious notation,

$$J_{11}(\phi, f_1(\epsilon)) = -\frac{k^2 \sin k(l_2 - l_1)}{2K(\epsilon) \sin k\epsilon} \int_{-\epsilon}^{\epsilon} \phi(z)\, dz. \tag{6.19}$$

Let us write

$$f_1(z) = \lim_{\epsilon \to 0} f_1(z, \epsilon), \tag{6.20}$$

so that

$$l_1 \le z \le 0: \quad f_1(z) = \sin k(l_1 - z)/\sin kl_1 , \tag{6.21a}$$

$$0 \le z \le l_2 : \quad f_1(z) = \sin k(l_2 - z)/\sin kl_2 . \tag{6.21c}$$

Proceeding to the limit $\epsilon \to 0$ in (6.19), we get (since $\phi(0) = 1$)

$$J_{11}(\phi, f_1) = kL\Gamma, \qquad \Gamma = \frac{\sin k(l_2 - l_1)}{\sin kl_1 \sin kl_2}. \tag{6.22}$$

Let us now put $f_1(z, \epsilon)$ for $f(z)$ in (6.14) and proceed to the limit $\epsilon \to 0$. In this limiting process, $f_1(z, \epsilon)$ and its first derivative remain finite in the whole range (l_1 , l_2), and so the contributions to J_{12} , J_{13} , J_2 from the range $(-\epsilon, \epsilon)$ vanish in the limit. Thus we get

$$Z = -\frac{i}{\pi c k^2} \frac{kL\Gamma + J_{12}(\phi, f_1) + J_{13}(\phi, f_1) + J_2(\phi, f_1)}{\int_{l_1}^{l_2} N(z) f_1(z)\, dz}, \tag{6.23}$$

where $f_1(z)$ is as in (6.21) and Γ as in (6.22); $L = -\ln(k^2 a^2)$.

This is accurate, and indeed holds for the general N of (5.4) as well as the more particular N of (5.9). The fact that $f_1(z)$ has a discontinuous first derivative at $z = 0$ creates no trouble. It is of course understood that in evaluating J_2 by (6.10) and J_{12} , J_{13} by (6.12), we are to put

$$\chi(z, z_0) = -\phi'(z) f_1'(z_0) + k^2 \phi(z) f_1(z_0). \tag{6.24}$$

On substituting for N from (5.9) in the denominator of (6.23), we obtain

$$Z = -\frac{i}{4\pi c k h} \{kL\Gamma + J_{12}(\phi, f_1) + J_{13}(\phi, f_1) + J_2(\phi, f_1)\},$$

$$h = \frac{\sin k\eta}{k\eta} + \tfrac{1}{2}\Gamma \frac{1 - \cos k\eta}{k\eta}. \tag{6.25}$$

This is the formula we shall use in the later work. It is accurate except for the approximation involved in (5.9).

Note: Do not confuse the range $(-\epsilon, \epsilon)$ with the gap $(-\eta, \eta)$. The former is merely a mathematical device, introduced to eliminate ϕ from the first term in the numerator

of (6.23). As a matter of fact, we shall make no further use of this ϵ; it has done its work in providing the formulae (6.23), (6.25).

7. The thin antenna. We cannot expect to get results for an antenna of general form without a considerable amount of calculation. But if the antenna is thin, the largeness of L may be used as a basis of approximation. In the present section we obtain the principal parts of the current and impedance for a thin antenna. This current is,

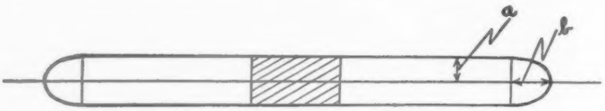

FIG. 13. Cylindrical antenna with spheroidal ends.

in fact, the familiar sinusoidal distribution. The present derivation of this distribution may be of interest because previous derivations have been by no means clear. Moreover, in the present method, it is not necessary to assume that the antenna is cylindrical; it is merely necessary that the radius R be small throughout. The ends of the antenna require no special treatment. Further, our formula for impedance contains a shape term. The method of the present section does not open up a process of successive approximations; that will be given in Sec. 10.

It will be well to mention here assumptions which will be introduced explicitly later:

The antenna is thin at the gap ($ka \ll 1$). (7.1)

The antenna is thin throughout ($kR(z) \ll 1$, $l_1 < z < l_2$). (7.2)

The gap is long compared with the radius ($a/\eta \ll 1$). (7.3)

The gap is short compared with the wave length ($k\eta \ll 1$). (7.4)

Obviously (7.2) contains (7.1).

We turn back to the exact formula (6.14), in which $f(z)$ is arbitrary. By (6.13) we have

$$J_{11} = L \int_{l_1}^{l_2} \{\phi''(z) + k^2\phi(z)\} f(z)\, dz. \tag{7.5}$$

We rearrange (6.14) in the form

$$\int_{l_1}^{l_2} \{\phi''(z) + k^2\phi(z) - i\pi ck^2 ZL^{-1}N(z)\} f(z)\, dz = -L^{-1}(J_{12} + J_{13} + J_2). \tag{7.6}$$

Now make the assumption (7.2). Then L is large and the right hand side of (7.6) is small of order L^{-1}, provided $\phi(z)$ and $\phi'(z)$ remain bounded as L tends to infinity. Since f is arbitrary except for the end conditions (6.6), it follows from (7.6) that

$$\phi''(z) + k^2\phi(z) - i\pi ck^2 ZL^{-1}N(z) = L^{-1}\mathcal{F}(z), \tag{7.7}$$

where $\mathcal{F}(z)$ is some finite function. Integration gives

$$\phi(z) = \alpha \cos kz + \beta \sin kz + i\pi ckZL^{-1} \int_0^z N(t) \sin k(z - t)\, dt + L^{-1}\mathcal{G}(z), \tag{7.8}$$

where $\mathcal{G}(z)$ is some finite function. This solution is subject to the three conditions (6.4), and if we knew $\mathcal{G}(z)$ we could find α, β, Z. But we do not know $\mathcal{G}(z)$, and can merely make use of the fact that the last term is of order L^{-1}.

Whatever $\mathcal{G}(z)$ may be, the three equations for α, β, Z are consistent provided

$$\begin{vmatrix} \sin kl_1 & \int_0^{l_1} N(t) \sin k(l_1 - t)\, dt \\ \sin kl_2 & \int_0^{l_2} N(t) \sin k(l_2 - t)\, dt \end{vmatrix} \neq 0. \tag{7.8a}$$

Let us make the assumption (7.3), so that we may use (5.9) for N. Then the above condition for consistency reads

$$(1 - \cos k\eta) \sin k(l_2 - l_1) + 2 \sin k\eta \sin kl_1 \sin kl_2 \neq 0. \tag{7.8b}$$

Assuming that $k\eta$, kl_1, kl_2 are such that this inequality is satisfied, we obtain α, β, Z from (6.4), and substitution in (7.8) gives accurately

$$\phi(z) = \phi_1(z) + L^{-1}\Omega(z), \tag{7.9}$$

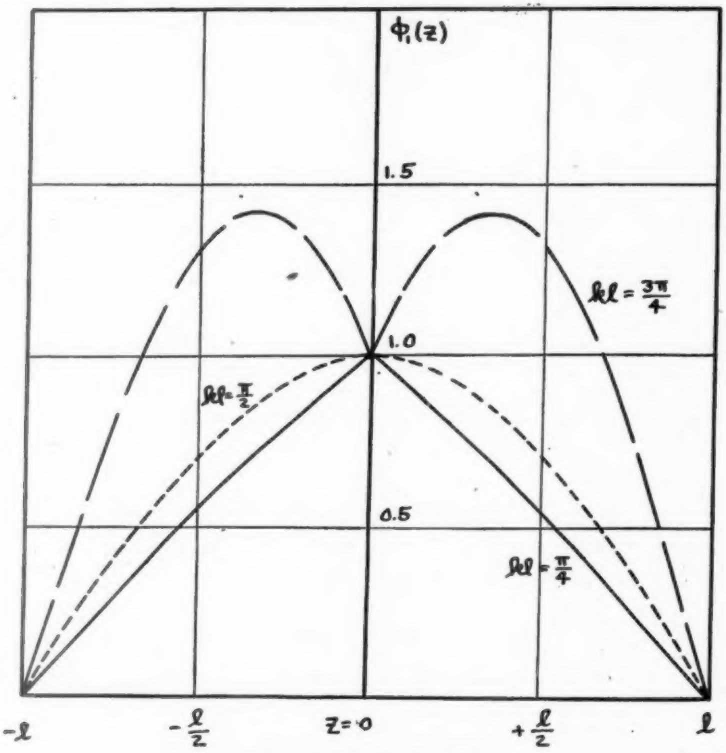

FIG. 14a. Infinitesimal gap at center.

where $\Omega(z)$ is an unknown finite function and $\phi_1(z)$ is given by

$$l_1 \leq z \leq -\eta : K(\eta)\phi_1(z) = -\sin kl_2 \sin k(l_1 - z), \tag{7.10a}$$

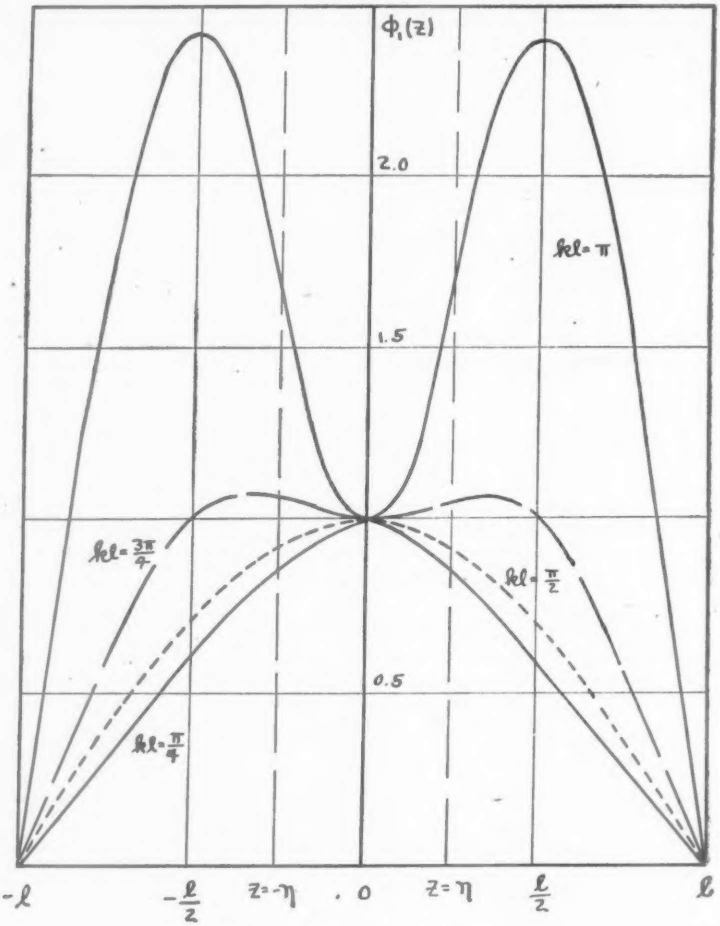

FIG. 14b. Finite gap ($\eta = l/4$) at center.

$$\begin{aligned} -\eta \le z \le \eta \ : K(\eta)\phi_1(z) &= K(\eta) \cos kz + \tfrac{1}{2} \sin k(l_1 + l_2) \sin kz \\ &\quad - \tfrac{1}{2} \operatorname{cosec} k\eta \sin k(l_2 - l_1)(1 - \cos kz), \end{aligned} \tag{7.10b}$$

$$\eta \le z \le l_2 \quad : K(\eta)\phi_1(z) = -\sin kl_1 \sin k(l_2 - z), \tag{7.10c}$$

$$K(\eta) = -\sin kl_1 \sin kl_2 - \tfrac{1}{2} \tan \tfrac{1}{2}k\eta \sin k(l_2 - l_1). \tag{7.11}$$

(These formulae should be compared with (6.15), (6.16). Note that $l_2 - l_1$ is the length of the antenna.)

Physically, $\phi_1(z)$ represents the principal part of the relative current for a thin antenna with a gap which is long compared with the radius of the antenna, and becomes

a better approximation the thinner the antenna becomes. Graphs of $\phi_1(z)$ are given in Figs. 14a–g (for discussion, see Appendix, p. 155). Outside the gap, $\phi_1(z)$ has the *sinusoidal distribution*, so basic in antenna theory.

To get the impedance, we substitute for ϕ from (7.9) in (6.25). This gives

$$Z = -\frac{i}{4\pi ckh}\{kL\Gamma + J_{12}(\phi_1, f_1) + J_{13}(\phi_1, f_1) + J_2(\phi_1, f_1) + L^{-1}[J_{12}(\Omega, f_1) + J_{13}(\Omega, f_1) + J_2(\Omega, f_1)]\}. \tag{7.12}$$

Here everything is known, except Ω.

Let us sum up:

For a thin antenna with a gap much longer than its radius, the relative current is given by (7.10), *with an error of order* L^{-1}, *and the impedance is*

$$Z_1 = -\frac{i}{4\pi ckh}\{kL\Gamma + J_{12}(\phi_1, f_1) + J_{13}(\phi_1, f_1) + J_2(\phi_1, f_1)\}, \tag{7.13}$$

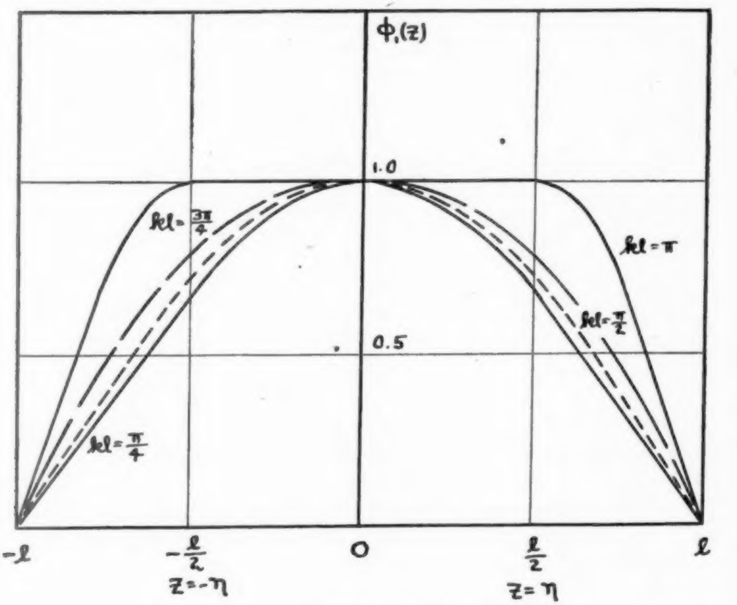

FIG. 14c. Finite gap ($\eta = l/2$) at center.

with an error of order L^{-1}. Here

$$L = -\ln(k^2a^2), \qquad a = \text{radius at gap}, \qquad k = 2\pi/\lambda,$$

$$h = \frac{\sin k\eta}{k\eta} + \tfrac{1}{2}\Gamma\,\frac{1 - \cos k\eta}{k\eta}, \qquad (-\eta < z < \eta) \text{ is gap}, \tag{7.14}$$

$$\Gamma = \frac{\sin k(l_2 - l_1)}{\sin kl_1 \sin kl_2} = \cot kl_1 - \cot kl_2\,.$$

We must remember that f_1 is given by (6.21) and does not depend on ϵ, whereas ϕ_1 is given by (7.10) and does depend on η.

On account of the assumed thinness of the antenna, we can simplify the expressions (6.10) for J_2 and (6.12) for J_{13}. In fact, we shall put $R = 0$ in J_2 and J_{13}, since the consequent error is of order ka, and so is negligible in comparison with L^{-1}. Thus we write

$$J_{12} = \int_{l_1}^{l_2} \chi(z, z) \ln \frac{a^2}{(R(z))^2} \, dz,$$

$$J_{13} = \int_{l_1}^{l_2} \chi(z, z) \ln [4k^2(l_2 - z)(z - l_1)] \, dz, \tag{7.15}$$

$$J_2 = \int_{l_1}^{l_2} \int_{l_1}^{l_2} | z - z_0 |^{-1} \{\chi(z, z_0) \exp ik \, | z - z_0 | - \chi(z, z)\} \, dz \, dz_0 ,$$

$$\chi(z, z_0) = -\phi_1'(z) f_1'(z_0) + k^2 \phi_1(z) f_1(z_0).$$

We note that the shape of the antenna is now involved only in J_{12}, and that there is no contribution to J_{12} from cylindrical parts of the antenna.

Putting $Z_1 = R_1 - iX_1$, the approximate resistance and reactance are (with an error of order L^{-1})

$$R_1 = \frac{1}{4\pi ckh} J_{22} ,$$

$$X_1 = \frac{1}{4\pi ckh} (kL\Gamma + J_{12} + J_{13} + J_{21}), \tag{7.16}$$

where

$$J_{21} + iJ_{22} = J_2 . \tag{7.17}$$

Thus the resistance is independent of shape.

8. Thin antenna with short gap. The calculation of the impedance from (7.13) is direct. The result will, of course, depend on the gap-length 2η, and will be very involved on account of the complexity of (7.10). Let us therefore, for simplicity, introduce the assumption (7.4); we consider the gap short compared with the wave-length, but still long compared with the radius of the antenna.

If $k\eta$ is small, $h = 1$ approximately. Further, by (7.10), we have approximately

$$l_1 \leq z \leq 0: \quad \phi_1(z) = \sin k(l_1 - z)/\sin kl_1 , \tag{8.1a}$$

$$0 \leq z \leq l_2 : \quad \phi_1(z) = \sin k(l_2 - z)/\sin kl_2 . \tag{8.1c}$$

(Note: the consistency condition (7.8b) becomes $\sin kl_1 \sin kl_2 \neq 0$ for small $k\eta$.) On comparison with (6.21), we see that $\phi_1 = f_1$. This greatly simplifies the work by introducing symmetry. We have

$$\chi(z, z_0) = -\phi_1'(z)\phi_1'(z_0) + k^2\phi_1(z)\phi_1(z_0), \tag{8.2}$$

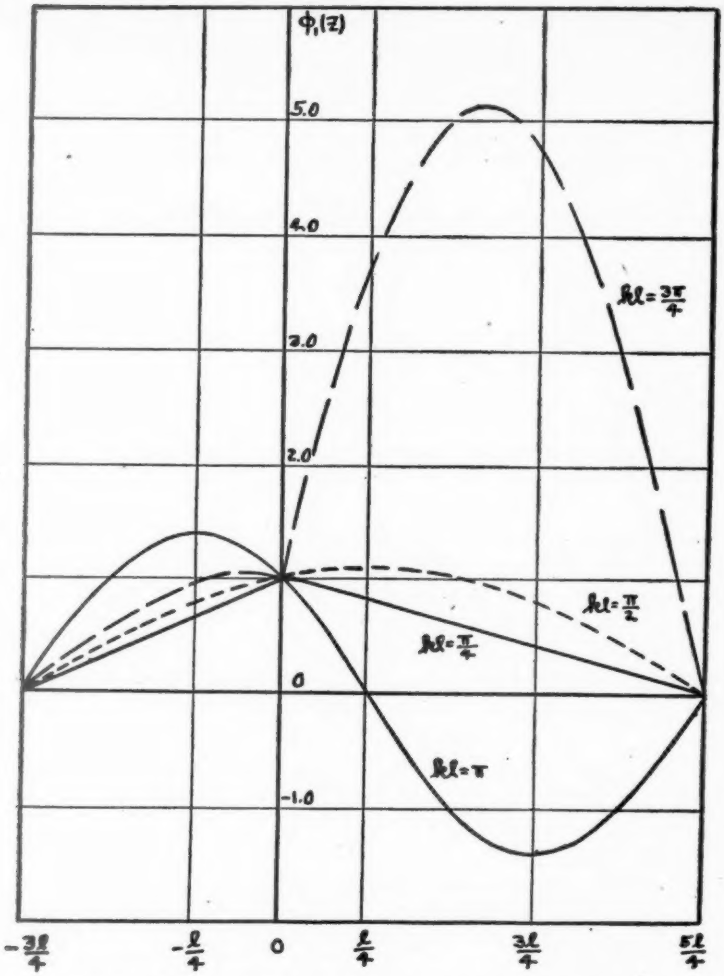

FIG. 14d. Infinitesimal gap at point dividing antenna in ratio 3:5.

and so

$$l_1 \leq z < 0 : \chi(z, z) = -k^2 \operatorname{cosec}^2 kl_1 \cos 2k(l_1 - z), \tag{8.3a}$$

$$0 < z \leq l_2 : \chi(z, z) = -k^2 \operatorname{cosec}^2 kl_2 \cos 2k(l_2 - z), \tag{8.3c}$$

$$z < 0, z_0 < 0 : \chi(z, z_0) = -k^2 \operatorname{cosec}^2 kl_1 \cos k(2l_1 - z - z_0), \tag{8.4aa}$$

$$z > 0, z_0 < 0, \text{ or } z < 0, z_0 > 0 : \qquad \chi(z, z_0) = -k^2 \operatorname{cosec} kl_1 \operatorname{cosec} kl_2 \cos k(l_1 + l_2 - z - z_0), \tag{8.4ac}$$

$$z > 0, z_0 > 0 : \chi(z, z_0) = -k^2 \operatorname{cosec}^2 kl_2 \cos k(2l_2 - z - z_0). \tag{8.4cc}$$

It is best to combine J_{13} and J_2 in (7.15). We have

$$\iint_{(\epsilon)} |z - z_0|^{-1}\chi(z, z)\, dz\, dz_0$$
$$= \int_{l_1}^{l_2} \chi(z, z) \ln[4k^2(l_2 - z)(z - l_1)]\, dz - 2\ln(2k\epsilon) \int_{l_1}^{l_2} \chi(z, z)\, dz. \tag{8.5}$$

Here $\iint_{(\epsilon)}$ means integration over the square (l_1, l_2) with omission of the strip $|z - z_0| < \epsilon$. Now

$$J_2 = \lim_{\epsilon \to 0} \iint_{(\epsilon)} |z - z_0|^{-1}\{\chi(z, z_0) \exp ik|z - z_0| - \chi(z, z)\}\, dz\, dz_0, \tag{8.6}$$

and so

$$J_{13} + J_2 = \lim_{\epsilon \to 0} \left[\iint_{(\epsilon)} |z - z_0|^{-1}\chi(z, z_0) \exp ik|z - z_0|\, dz\, dz_0 \right.$$
$$\left. + 2\ln(2k\epsilon) \int_{l_1}^{l_2} \chi(z, z)\, dz \right]. \tag{8.7}$$

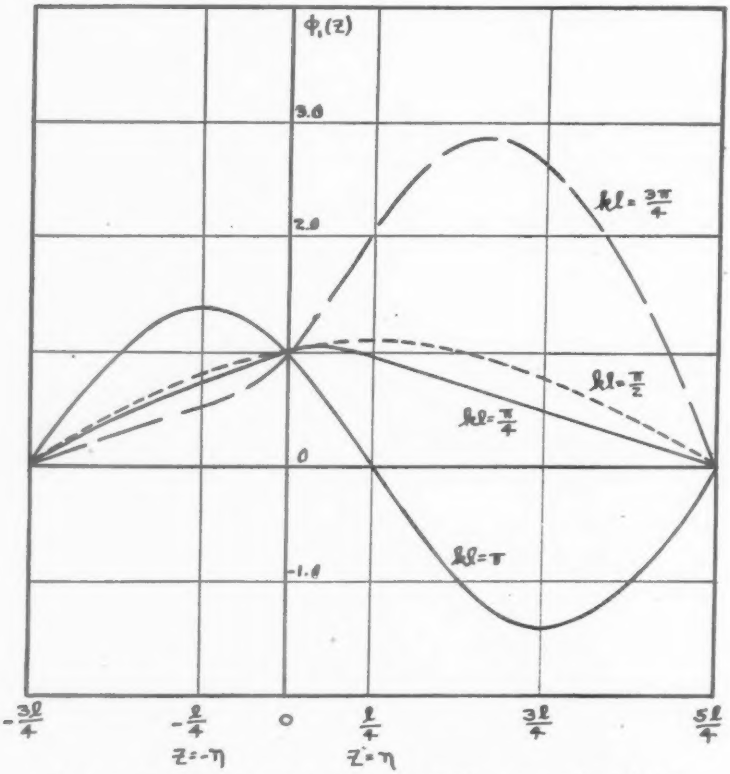

FIG. 14e. Finite gap ($\eta = l/4$) with center dividing antenna in ratio 3:5.

This double integral is best evaluated by rotating the z, z_0 axes through half a right angle in the plane of the integral. It is convenient to introduce the function

$$\Phi(x) = \int_0^x \frac{e^{it} - 1}{t}\,dt = \mathrm{Ci}\,x + i\,\mathrm{Si}\,x - \ln \gamma x, \qquad \ln \gamma = 0.5772. \tag{8.8}$$

We find

$$\begin{aligned}(J_{13} + J_2)/k = (c_2 - c_1) \ln \frac{(l_2 - l_1)^2}{4k^2 l_1^2 l_2^2} &\\ + \Phi(4kl)\{c_2 - c_1 - i(c_1c_2 + 1)\}&\\ - \Phi(2kl_2)\{c_2 - c_1 + ic_2(c_2 - c_1)\}&\\ - \Phi(-2kl_1)\{c_2 - c_1 - ic_1(c_2 - c_1)\},&\end{aligned} \tag{8.9}$$

$$c_2 = \cot kl_2\,, \qquad c_1 = \cot kl_1\,, \qquad 2l = l_2 - l_1 = \text{length of antenna.}$$

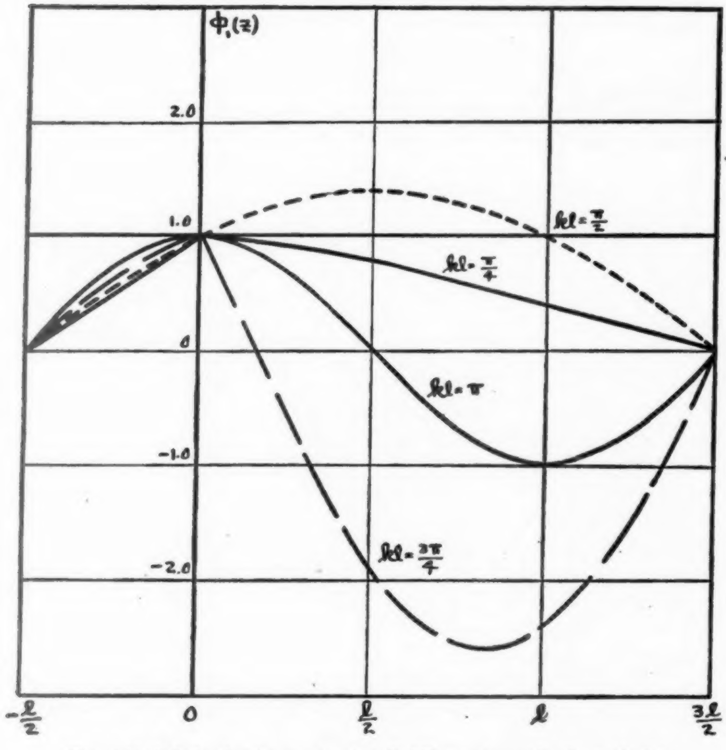

FIG. 14f. Infinitesimal gap at point dividing antenna in the ratio 1:3.

When we substitute from (8.9) in (7.13), we get the following expression for *the principal part of the impedance of a thin antenna,* expressed in ohms (1 Heaviside unit = 120 πc ohms):

$$
\begin{aligned}
Z_1 &= R_1 - iX_1 \\
&= 30 \frac{\sin 2kl}{\sin k \mid l_1 \mid \sin kl_2} \left\{ 2i \ln \left| \frac{l_1 l_2}{al} \right| - \Phi(4kl)(i - \cot 2kl) \right. \\
&\quad + \Phi(2k \mid l_1 \mid)(i - \cot k \mid l_1 \mid) \\
&\quad \left. + \Phi(2kl_2)(i - \cot kl_2) \right\} - iX_s ,
\end{aligned}
\tag{8.10}
$$

where

$$
\begin{aligned}
X_s = 30J_{12}/k = &-30k \operatorname{cosec}^2 kl_1 \int_{l_1}^{0} \cos 2k(l_1 - z) \ln \frac{a^2}{(R(z))^2}\, dz \\
&-30k \operatorname{cosec}^2 kl_2 \int_{0}^{l_2} \cos 2k(l_2 - z) \ln \frac{a^2}{(R(z))^2}\, dz.
\end{aligned}
\tag{8.11}
$$

Here $z = l_1$, $z = l_2$ are the ends of the antenna ($l_1 < 0 < l_2$); the gap is short and at $z = 0$; $2l$ is the length of the antenna; a is the radius at the gap, and $R(z)$ the radius at the general point; $k = 2\pi/\lambda$, where λ is the free wave length.

If we take the gap at the middle, then $-l_1 = l_2 = l$, and (8.10) becomes

$$
\begin{aligned}
Z_1 = 60 \cot kl\{2i \ln(l/a) &- \Phi(4kl)(i - \cot 2kl) \\
&+ 2\Phi(2kl)(i - \cot kl)\} - iX_s .
\end{aligned}
\tag{8.12}
$$

Except for the shape term X_s, this agrees with the formula given by Brown and King[1] using the method of Labus.[2] There is also agreement with the principal part of Schelkunoff's formula[3] and with the formula of Hallén.[4] Schelkunoff's treatment of the influence of shape is difficult to follow. Hallén includes a shape term in his equation (26), but later specializes to a cylindrical antenna, so that a formula such as our (8.11) does not occur explicitly in his equation (39). Owing to the inadequate treatment of the gap in the work of Hallén and Schelkunoff, the validity of their higher approximations is open to question. It must be remembered that an error of the order L^{-1} is admitted in our formulae (8.10), (8.12).

In (8.12) it is not assumed that the antenna has $z = 0$ for equatorial plane of symmetry. Deviation from this symmetry influences X_s, but not the other terms.

Let us consider an antenna with the gap at the center and total length nearly $\frac{1}{2}\lambda$, so that

$$kl = \tfrac{1}{2}\pi + \epsilon, \tag{8.13}$$

where ϵ is small. Then, approximately, with errors of orders ϵ^3 and ϵ^2 respectively,

$$\cot kl = -\epsilon, \qquad \cot kl \cot 2kl = -\tfrac{1}{2}. \tag{8.14}$$

---

[1]G. H. Brown and R. King, Proc. I. R. E. **22**, 457-480 (1934).

[2]J. Labus, Hochfrequenztechnik und Elektroakustik **41**, 17-23 (1933).

[3]S. A. Schelkunoff, Proc. I. R. E. **29**, 493-521 (1941).

[4]E. Hallén, Nova Acta Reg. Soc. Sci. Upsaliensis **11**, No. 4 (1939).

Then we may neglect parts of (8.12) and write

$$\begin{aligned} Z_1 &= R_1 - iX_1 = -120i\epsilon \ln \frac{l}{a} - 30\Phi(2\pi) - iX_s\,, \\ R_1 &= 30(\log 2\pi\gamma - \text{Ci}\, 2\pi) = 73.13, \\ X_1 &= 120\epsilon \ln \frac{l}{a} + X_s + 30\,\text{Si}\, 2\pi, \\ &= 120\epsilon \ln \frac{l}{a} + X_s + 42.54. \end{aligned} \tag{8.15}$$

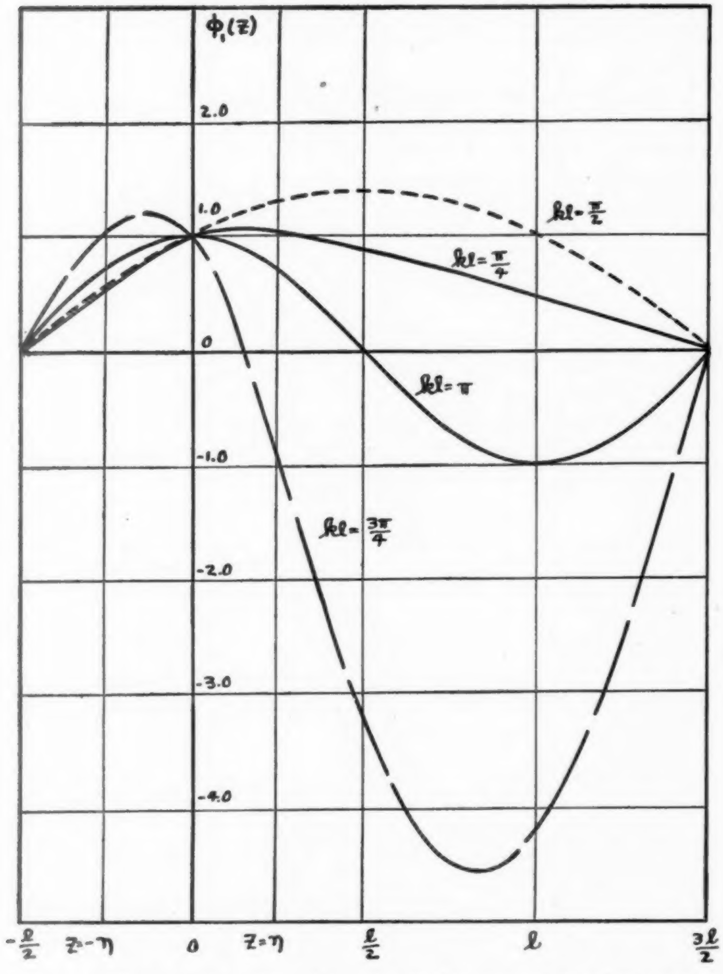

FIG. 14g. Finite gap ($\eta = l/4$) with center dividing antenna in the ratio 1:3.

The case where the antenna is approximately of total length $\frac{1}{2}\lambda$, but with the gap not at the center, is also of interest. Again we have (8.14), and substitution in (8.10) gives approximately

$$
\begin{aligned}
Z_1 &= -30 \operatorname{cosec}^2 kl_2 \left\{ \Phi(2\pi) + 4i\epsilon \ln \left| \frac{l_1 l_2}{al} \right| \right\} - iX_s\,, \\
R_1 &= 30 \operatorname{cosec}^2 kl_2 (\ln 2\pi\gamma - \mathrm{Ci}\, 2\pi) = 73.13 \operatorname{cosec}^2 kl_2\,, \\
X_1 &= 120\epsilon \operatorname{cosec}^2 kl_2 \ln \left| \frac{l_1 l_2}{al} \right| + X_s + 42.54 \operatorname{cosec}^2 kl_2\,.
\end{aligned}
\tag{8.16}
$$

Thus, by moving the gap away from the center of the antenna, we increase the resistance, but the tuning of the antenna to make $X_1 = 0$ is more difficult, because the derivative of X_1 with respect to ϵ is greater.

We see that the problem of matching the antenna to a coaxial line, as far as reactance is concerned, depends on the shape term X_s in an important way. This term is discussed in the next section.

9. The shape term in the reactance. Let us consider the term X_s, given in (8.11). For simplicity, let us assume that the gap is at the center of the antenna and that the antenna has an equatorial plane of symmetry ($z = 0$). Then (8.11) reads

$$X_s = -60k \operatorname{cosec}^2 kl \int_0^l \cos 2k(l - z) \ln \frac{a^2}{(R(z))^2}\, dz, \tag{9.1}$$

$$2l = \text{length of antenna.}$$

We note that X_s receives no contribution from cylindrical portions of the antenna.

We can now settle the vexed question of contribution from the ends of a cylindrical antenna, by supposing the antenna to be a cylinder ($R = a$), terminated by spheroids of semi-axis b (Fig. 13, p. 138). Then on the ends

$$\frac{(R(z))^2}{a^2} + \frac{(z - l + b)^2}{b^2} = 1. \tag{9.2}$$

No assumption is made at first about the magnitude of b. Equation (9.1) gives

$$X_s = 60k \operatorname{cosec}^2 kl \int_{l-b}^{l} \cos 2k(l - z) \ln[1 - (z - l + b)^2/b^2]\, dz. \tag{9.3}$$

Since the logarithm breaks into the sum of two logarithms, this integral is easily evaluated in terms of Ci and Si functions; we find

$$
\begin{aligned}
X_s = 30 \operatorname{cosec}^2 kl\{&\sin 4kb(\mathrm{Ci}\, 4kb - \mathrm{Ci}\, 2kb) \\
&- \cos 4kb\, \mathrm{Si}\, 4kb - (1 - \cos 4kb)\, \mathrm{Si}\, 2kb\}.
\end{aligned}
\tag{9.4}
$$

If kb is small, this approximates to

$$X_s = 120\, kb.\ \operatorname{cosec}^2 kl.\ (\ln 2 - 1) \text{ ohms.} \tag{9.5}$$

Since this tends to zero with kb, we see that *there is no contribution to reactance from the ends of a cylindrical antenna cut off square at the ends.* (Of course, "contribution from

the ends" implies some mathematical division of reactance into "contributions" of various sorts; our statement refers to the division we have made in using the integral (9.1).) In fact, $X_s = 0$ to our order of approximation if the ends are rounded for a length b comparable to the radius a of the cylindrical part even for non-central gap.

Spheroidal antenna. The impedance of a spheroidal antenna can be calculated accurately by means of spheroidal functions.[5] However for a thin spheroid, the present method may be used. Let us take the gap at the center. Then

$$(R(z))^2/a^2 = (l^2 - z^2)/l^2, \tag{9.6}$$

and so by (9.1)

$$\begin{aligned} X_s &= 60k \operatorname{cosec}^2 kl \int_0^l \cos 2k(l - z) \ln[(l^2 - z^2)/l^2]\, dz \\ &= 30 \operatorname{cosec}^2 kl\{\sin 4kl(\mathrm{Ci}\, 4kl - \mathrm{Ci}\, 2kl) \\ &\qquad - \cos 4kl(\mathrm{Si}\, 4kl - \mathrm{Si}\, 2kl) - \mathrm{Si}\, 2kl\}. \end{aligned} \tag{9.7}$$

For $l = \lambda/4$, $kl = \frac{1}{2}\pi$, we have

$$X_s = -30\, \mathrm{Si}\, 2\pi. \tag{9.8}$$

Referring to (8.15) with $\epsilon = 0$, we see that $X_1 = 0$; *the reactance of a thin spheroidal half-wave antenna (with the gap in middle) is zero.* This fact is mentioned by Schelkunoff (loc. cit.), but the reason for this statement is not clear.

Conical antenna. For a symmetrical thin conical antenna fed at the vertex, we put

$$R(z) = \beta z \tag{9.9}$$

where β is the semi-angle of the cone. Equation (9.1) gives at once for the shape term

$$X_s = 120 \cot kl \ln \frac{\beta}{ka} + 60. \tag{9.10}$$

For the approximation to be valid, we should take β of the same order as ka.

Graphs of impedance are given in Fig. 15 (see Appendix).

10. Successive approximations. The method of Sections 7 and 8 gives the principal part of the impedance for a thin antenna, but it does not open up a method of successive approximations. To get such a method, let us return to the exact integral equation (6.3) and write it in a slightly different notation as follows:

$$-\int_{l_1}^{l_2} \frac{\partial^2 \psi(t, z)}{\partial z\, \partial t} \phi(t)\, dt + k^2 \int_{l_1}^{l_2} \psi(t, z)\phi(t)\, dt = i\pi c k^2 Z N(z), \qquad (l_1 < z < l_2). \tag{10.1}$$

Let us introduce the integration operator S such that

$$Sf(z) = \int_0^z f(t)\, dt. \tag{10.2}$$

[5] J. A. Stratton and L. J. Chu, J. Applied Physics **12**, 241-248 (1941); L. Infeld, Q. Appl. Math. **5**, 113-132 (1947).

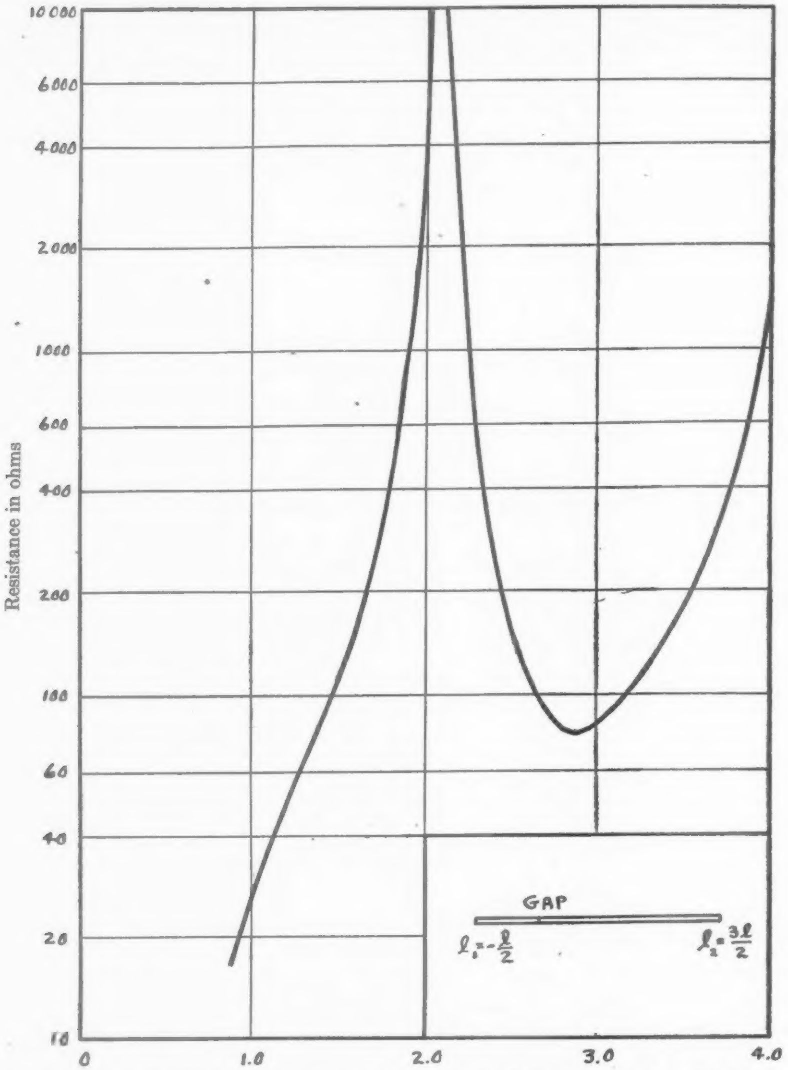

FIG. 15a. Resistance of thin antenna with infinitesimal gap at point of quadrisection.

Then

$$
\begin{aligned}
-S \int_{l_1}^{l_2} \frac{\partial^2 \psi(t, z)}{\partial z\, \partial t} \phi(t)\, dt &= -S \frac{d}{dz} \int_{l_1}^{l_2} \frac{\partial \psi(t, z)}{\partial t} \phi(t)\, dt \\
&= -\int_{l_1}^{l_2} \frac{\partial \psi(t, z)}{\partial t} \phi(t)\, dt + \int_{l_1}^{l_2} \frac{\partial \psi(t, 0)}{\partial t} \phi(t)\, dt \qquad (10.3) \\
&= -\int_{l_1}^{l_2} \psi(t, z) \phi'(t)\, dt - A,
\end{aligned}
$$

on account of (6.4). Here A is a constant,

$$A = \int_{l_1}^{l_2} \psi(t, 0)\phi'(t)\, dt. \tag{10.4}$$

Now we operate on (10.1) with S^2; this gives

$$S \int_{l_1}^{l_2} \psi(t, z)\phi'(t)\, dt - Az + k^2 S^2 \int_{l_1}^{l_2} \psi(t, z)\phi(t)\, dt = i\pi c k^2 Z S^2 N(z). \tag{10.5}$$

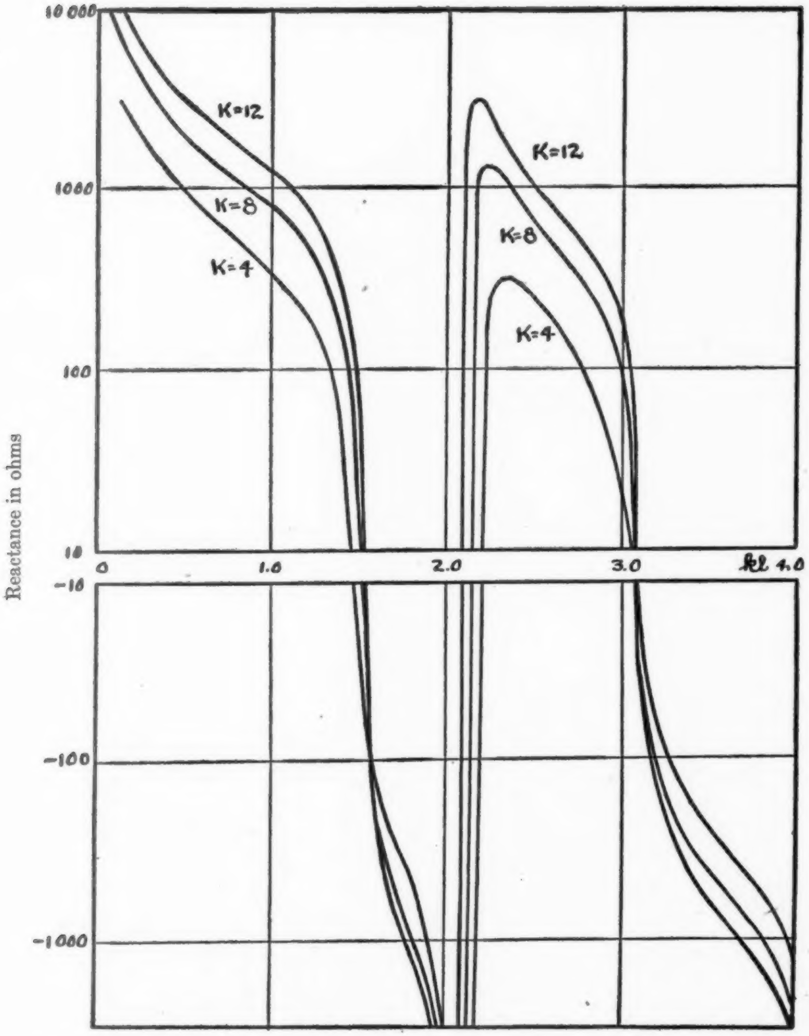

FIG. 15b. Reactance of a thin cylindrical antenna with infinitesimal gap at point of quadrisection.

To treat these integrals, consider the operator B, defined by

$$\begin{aligned} Bf(z) &= S\int_{l_1}^{l_2} \psi(t, z)f(t)\,dt \\ &= \int_{l_1}^{l_2} f(t)\,dt \int_0^z \psi(t, y)\,dy \\ &= B_1 f(z) + B_2 f(z), \end{aligned} \tag{10.6}$$

where

$$\begin{aligned} B_1 f(z) &= \int_{l_1}^{l_2} f(t)\,dt \int_0^z \frac{dy}{r(t, y)}, \qquad [r(t, y)]^2 = [R(t)]^2 + (y - t)^2, \\ B_2 f(z) &= \int_{l_1}^{l_2} f(t)\,dt \int_0^z \frac{\exp ikr(t, y) - 1}{r(t, y)}\,dy. \end{aligned} \tag{10.7}$$

Then

$$\begin{aligned} B_1 f(z) &= \int_{l_1}^{l_2} \ln \frac{z - t + \{[R(t)]^2 + (z - t)^2\}^{1/2}}{-t + \{[R(t)]^2 + t^2\}^{1/2}} f(t)\,dt \\ &= \left(\int_{l_1}^{z} + \int_z^{l_2}\right) \ln k[z - t + \{[R(t)]^2 + (z - t)^2\}^{1/2}]f(t)\,dt \\ &\qquad - \left(\int_{l_1}^{0} + \int_0^{l_2}\right) \ln k[-t + \{[R(t)]^2 + t^2\}^{1/2}]f(t)\,dt \\ &= \left(\int_{l_1}^{z} - \int_z^{l_2}\right) \ln k[\,|z - t| + \{[R(t)]^2 + (z - t)^2\}^{1/2}]f(t)\,dt \\ &\qquad - \left(\int_{l_1}^{0} - \int_0^{l_2}\right) \ln k[\,|t| + \{[R(t)]^2 + t^2\}^{1/2}]f(t)\,dt \\ &\qquad + \int_z^{l_2} \ln[kR(t)]^2 f(t)\,dt - \int_0^{l_2} \ln[kR(t)]^2 f(t)\,dt \\ &= B_3 f(z) + B_4 f(z) - \ln(k^2a^2)Sf(z), \end{aligned} \tag{10.8}$$

where

$$\begin{aligned} B_3 f(z) &= \left(\int_{l_1}^{z} - \int_z^{l_2}\right) \ln k[\,|z - t| + \{[R(t)]^2 + (z - t)^2\}^{1/2}]f(t)\,dt \\ &\qquad - \left(\int_{l_1}^{0} - \int_0^{l_2}\right) \ln k[\,|t| + \{[R(t)]^2 + t^2\}^{1/2}]f(t)\,dt, \\ B_4 f(z) &= \int_0^z \ln[a/R(t)]^2 f(t)\,dt. \end{aligned} \tag{10.9}$$

In the notation of (10.6), (10.5) reads

$$B\phi'(z) - Az + k^2 SB\phi(z) = i\pi ck^2 ZS^2 N(z), \tag{10.10}$$

or, since $S\phi'(z) = \phi(z) - 1$,

$$L[\phi(z) - 1 + k^2S^2\phi(z)] + F\phi(z) - Az = i\pi ck^2ZS^2N(z), \tag{10.11}$$

where $L = -\ln k^2a^2$, and F is the operator

$$F = (B_2 + B_3 + B_4)D + k^2S(B_2 + B_3 + B_4). \tag{10.12}$$

Here D is the derivative symbol. Equation (10.11) gives

$$(1 + k^2S^2)\phi(z) = 1 + L^{-1}Az - L^{-1}F\phi(z) + L^{-1}i\pi ck^2ZS^2N(z). \tag{10.13}$$

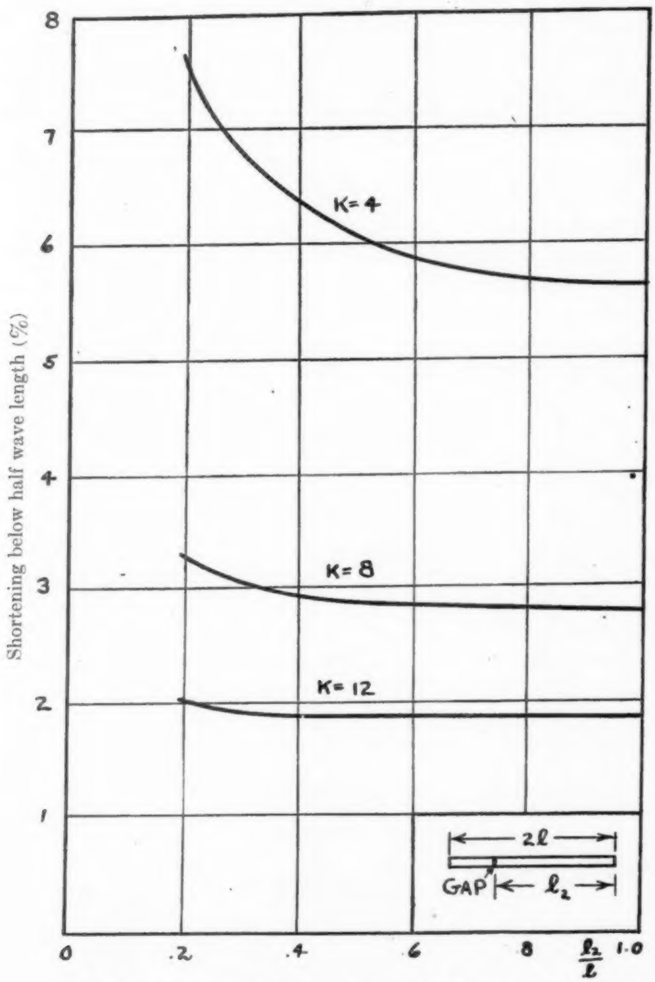

FIG. 15c. Resonant length of a thin cylindrical antenna with flat ends and infinitesimal gap at different points.

Now the operator P inverse to $1 + k^2S^2$ is easily found. It is

$$(1 + k^2S^2)^{-1}f(z) = Pf(z) = f(z) - k \int_0^z \sin k(z - t)f(t)\,dt. \tag{10.14}$$

This means that for an arbitrary function $f(z)$ we have $P(1 + k^2S^2)f(z) = (1 + k^2S^2)Pf(z) = f(z)$; this is easily verified.

In particular, we have

$$P.1 = \cos kz, \qquad Pz = k^{-1} \sin kz. \tag{10.15}$$

Thus (10.13) gives

$$\phi(z) = \cos kz + L^{-1}Ak^{-1} \sin kz - L^{-1}PF\phi(z) + L^{-1}i\pi ck^2ZPS^2N(z). \tag{10.16}$$

This is a *transform of the basic integral equation* (6.3). It is exact because in deriving it, we have not actually used the approximation (5.9). With (10.16) we are to associate the boundary conditions (6.4). The first of these is satisfied automatically. The others give

$$\begin{aligned} &\cos kl_1 + L^{-1}Ak^{-1} \sin kl_1 - L^{-1}T_{l_1}PF\phi(z) + L^{-1}i\pi ck^2ZT_{l_1}PS^2N(z) = 0, \\ &\cos kl_2 + L^{-1}Ak^{-1} \sin kl_2 - L^{-1}T_{l_2}PF\phi(z) + L^{-1}i\pi ck^2ZT_{l_2}PS^2N(z) = 0. \end{aligned} \tag{10.17}$$

Here T_{l_1}, T_{l_2} are substitution operators, meaning "put $z = l_1$, or $z = l_2$, finally." We now eliminate A and Z from (10.16), (10.17); this gives for $\phi(z)$ the equation

$$\begin{vmatrix} \phi(z) - \cos kz + L^{-1}PF\phi(z) & \sin kz & PS^2N(z) \\ -\cos kl_1 + L^{-1}T_{l_1}PF\phi(z) & \sin kl_1 & T_{l_1}PS^2N(z) \\ -\cos kl_2 + L^{-1}T_{l_2}PF\phi(z) & \sin kl_2 & T_{l_2}PS^2N(z) \end{vmatrix} = 0. \tag{10.18}$$

Remembering that $N(z)$ is known, the plan of solving by successive approximations is now obvious. We are to substitute $\phi = \phi_1$ in the PF column, ϕ_1 being some initial approximation, and solve, obtaining ϕ_2. Then ϕ_2 is to be substituted in the PF column, and the equation solved, giving ϕ_3; and so on. At any stage, we might get Z from (10.17) but it seems probable that a better value will be given by (6.23) or (6.25).

Under the assumptions (7.1), (7.3) we have (5.9), and hence we can calculate $PS^2N(z)$. In general

$$\begin{aligned} Pf(z) &= f(z) - k \int_0^z f(t) \sin k(z - t)\,dt \\ &= f(0) \cos kz + \int_0^z Df(t)\cdot \cos k(z - t)\,dt \\ &= f(0) \cos kz + k^{-1}f'(0) \sin kz + k^{-1} \int_0^z D^2f(t)\cdot \sin k(z - t)\,dt. \end{aligned} \tag{10.19}$$

Hence

$$PS^2N(z) = k^{-1} \int_0^z N(t) \sin k(z - t)\,dt, \tag{10.20}$$

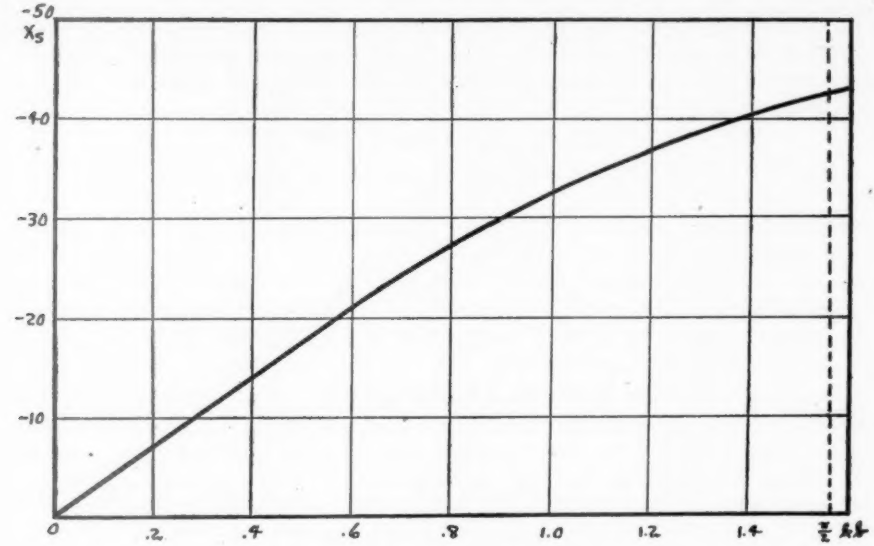

FIG. 15d. Shape term in the reactance of a thin cylindrical half wave antenna terminated by spheroids of semi-axis b (center gap).

and substituting from (5.9) we have

$$l_1 \leq z \leq -\eta : PS^2N(z) = -2k^{-3}\eta^{-1}[\cos k(z + \eta) - \cos kz], \tag{10.21a}$$

$$-\eta \leq z \leq \eta : PS^2N(z) = -2k^{-3}\eta^{-1}(1 - \cos kz), \tag{10.21b}$$

$$\eta \leq z \leq l_2 : PS^2N(z) = -2k^{-3}\eta^{-1}[\cos k(z - \eta) - \cos kz]. \tag{10.21c}$$

It is natural to take as first approximation $\phi_1(z)$, the function obtained by deleting the PF terms from (10.18). This $\phi_1(z)$ is precisely that given by (7.10). Then the second approximation $\phi_2(z)$ will be given by

$$\begin{vmatrix} \phi_2(z) - \cos kz + L^{-1}PF\phi_1(z) & \sin kz & PS^2N(z) \\ - \cos kl_1 + L^{-1}T_{l_1}PF\phi_1(z) & \sin kl_1 & T_{l_1}PS^2N(z) \\ - \cos kl_2 + L^{-1}T_{l_2}PF\phi_1(z) & \sin kl_2 & T_{l_2}PS^2N(z) \end{vmatrix} = 0, \tag{10.22}$$

and the higher approximations by similar formulae.

APPENDIX: NOTES ON FIGURES 14 AND 15.

Fig. 14. Relative current in a thin antenna. The following graphs are drawn from equations (7.10), which were obtained on the following assumptions:

(*i*) the tangential electric field is constant over the gap;
(*ii*) the antenna is very thin, but not necessarily cylindrical with flat ends;
(*iii*) when the gap is infinitesimal (as in Figs. 14 a, d, f) its length is still much greater than the infinitesimal radius of the antenna.

The ordinate $\phi_1(z)$ in each graph is the principal part of the relative current, $I(z)/I(0)$. The abscissa z represents position on the antenna. The ends of the antenna are $z = l_1$, $z = l_2$, and the ends of the gap are $z = -\eta$, $z = \eta$, so that $z = 0$ is the center of the gap. Also

$$k = 2\pi/\lambda, \qquad \lambda = \text{wave length},$$

$$2l = l_2 - l_1 = \text{total length of antenna}.$$

In each figure graphs are drawn for various values of kl up to $kl = \pi$.

The following points are of interest:

1) When the gap is finite, the derivative of $\phi_1(z)$ is continuous; when the gap is infinitesimal, the derivative is discontinuous at the gap, unless $kl = \frac{1}{2}\pi$ or π.
2) The curve for $kl = \frac{1}{2}\pi$ (that is, $2l = \frac{1}{2}\lambda$) is always a single sine curve, whether the gap is finite or infinitesimal.
3) Certain curves are not shown, because they go to infinity. This means that $I(0) = 0$ and the impedance is infinite (to this order of approximation). This occurs for $kl = \pi$ (that is, $2l = \lambda$) in Fig. 14a. This infinity disappears when the gap is widened in Fig. 14b. This does not mean that the widening of the gap eliminates infinite impedance; in Fig. 14b there is infinite impedance for $kl = 8\pi/7$, since this makes $K(\eta)$ vanish in (7.11). However, when the gap is finite, it is questionable whether the impedance is correctly defined (for matching purposes) by $Z = V/I(0)$.

   For infinitesimal eccentric gaps, we get infinite impedance for $kl = 4\pi/5$ in Fig. 14d and $kl = 2\pi/3$ in Fig. 14f.
4) Fig. 14c has a remarkable feature: the current is constant in the gap for $kl = \pi$.

Fig. 15. Impedance of a thin antenna. These graphs are based on equations (8.10) and (9.1).

Fig. 15a shows the resistance of a thin antenna plotted against kl. The gap is infinitesimal and situated at the point of quadrisection. It is interesting to compare this curve with those given by King[6] and Schelkunoff (loc. cit.) for an antenna with central gap. The effect of moving the gap from the center to the point of quadrisection is to change the point of great or infinite resistance from $kl = \pi$ to $kl = 2\pi/3$. This comes from the factor $\sin kl_2$ in the denominator in (8.10). Further, the resistance of a half-wave antenna ($kl = \frac{1}{2}\pi$) is changed by this shift of gap from about 70 to about 140 ohms.

Fig. 15b shows the reactance of a thin cylindrical antenna plotted against kl. As in the case of Fig. 15a, the gap is infinitesimal and at the point of quadrisection. Graphs are plotted for several values of the thickness parameter $K = \ln (l/a)$, where $2l$ is the length of the antenna and a its radius. The reactance vanishes not only in the neighbourhood of $kl = \frac{1}{2}\pi$ and $kl = \pi$, but also in the neighbourhood of $kl = 2\pi/3 = 2.09$.

The resonant length of a thin cylindrical antenna (i.e. the length making the reactance vanish) is a little less than half a wave length. The shortening below the half wave length is a function of the position of the gap. This dependence is shown graphically in Fig. 15c.

Fig. 15d shows the effect of rounding the ends of a cylindrical antenna, as in Fig. 13. The antenna is half a wave length long and the gap is infinitesimal and at the center. Flat ends correspond to $kb = 0$ and a completely spheroidal antenna to $kb = \frac{1}{2}\pi$.

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[6] L. V. King, Phil. Trans. Roy. Soc. (A) **236**, 381-422 (1937).

THE RADIATION AND TRANSMISSION PROPERTIES OF A PAIR OF SEMI-INFINITE PARALLEL PLATES—I*

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1. Introduction. We are concerned here with the following problem. A plane monochromatic electromagnetic wave is incident upon a pair of semi-infinite parallel metallic plates of zero thickness and perfect conductivity (see Fig. 1 for a side view). The edges of the plates are infinite straight lines which are parallel to the y axis of an xyz rectangular coordinate system. (The y axis is perpendicular to the plane of the paper in Fig. 1). The plates extend indefinitely in the direction of the positive z axis and are spaced a units apart. It is assumed, as in CHI[1], that the electric field of the incident wave has only one component, namely the one which is parallel to the y axis. Since the incident electric field is independent of y, and the boundary conditions on the plates are fulfilled independently of y, no other components of the electric field will be excited. There will be two components of the magnetic field; these in turn may be derived from the single component of the electric field through the Maxwell equations. The angle θ, the direction of the propagation vector of the incident wave, is measured with respect to the positive z axis.

We have just described the manner in which Fig. 1 is excited from free space. It is now necessary to indicate the mode of excitation which the parallel plate region can sustain. We assume that for $z \gg 0$, $0 \leq x \leq a$, the y component of the electric field is asymptotic to $(\rho_1 e^{i\kappa z} + \rho_2 e^{-i\kappa z}) \sin (\pi x/a)$. That is, the parallel plate region can sustain a mode which is consistent with the polarization which we consider here. It is to be understood that there are no other means of excitation in the finite part of the xyz space. κ is the propagation constant in the parallel plate region and is equal to $(k^2 - (\pi/a)^2)^{1/2}$, where $k = 2\pi/\lambda$ and λ is the free space wave length. In order that a

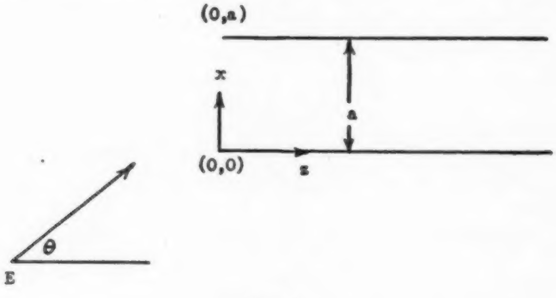

FIG. 1.

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[1]Carlson and Heins, *The reflection of an electromagnetic plane wave by an infinite set of plates I*, this Quarterly, **4**, 313-329 (1947). Hereafter we shall refer to this as CHI. We employ here the same form of the Maxwell equations which were used in CHI. The time dependence is taken as exp $(-ikct)$ where c is the velocity of light.

single mode propagate in the parallel plate region, it is necessary to assume further that $\frac{1}{2} < a/\lambda < 1$. The constant ρ_1 is the amplitude of the wave going to the right in the parallel plate region, while the ρ_2 is the amplitude of the wave going to the left in the same region.

This structure may be viewed as a two dimensional antenna in the sense that it can receive and transmit energy. The problem can be formulated mathematically as a pair of simultaneous integral equations of the faltung type which are closely related to those of the Wiener-Hopf type.[2] The unknown functions in these integral equations are the surface current densities on the plates. The solution of these equations will give us the functional form of the current densities, their asymptotic form for $z \to 0^+$, $z \to \infty$, as well as the relation between the amplitudes of the various waves. We shall divide this problem into two parts. In the first part we shall assume excitation from free space. Then the parallel plate region sustains the wave travelling to the right, since there are no obstacles in the parallel plate region which would give rise to a wave travelling to the left. In this case, we find the magnitude as well as the phase of the parallel plate wave which is travelling to the right. In the second part, we shall assume that the parallel plate region has been excited. Here we shall find the reflection coefficient, that is, the ratio of ρ_1/ρ_2. This second problem breaks down into a single integral equation due to the presence of a symmetry in the field components. The first problem we treat considers the parallel plate region as a receiving antenna, while the second one considers it as a transmitting antenna. We shall see that their properties are not completely independent.

The formulation of the pair of simultaneous integral equations which we have just mentioned can be carried out by the same method employed by Carlson and Heins (CHI). An application of Green's integral theorem in two dimensions with a free space Green's function as a kernel gives us the y component of the electric field in terms of the surface current density on each plate. Thus if $E_y(x, z)$ is the y component of the electric field, and $I_0(z)$ and $I_1(z)$ the surface current densities on the lower and upper plates respectively, we have the following relation

$$\begin{aligned} E_y(x, z) &= E_y^{inc}(x, z) \\ &+ \frac{i}{4}\int_0^\infty \{I_0(z')H_0^{(1)}[k(x^2 + (z - z')^2)^{1/2}] + I_1(z')H_0^{(1)}[k((x - a)^2 + (z - z')^2)^{1/2}]\}\, dz', \end{aligned} \tag{1.1}$$

where $H_0^{(1)}$ is the Hankel function of the first kind and $E_y^{inc}(x, z) = \exp[ik(x \sin\theta + z\cos\theta)]$. The boundary conditions on $E_y(x, z)$ give us the simultaneous integral equations. Indeed, since $E_y(x, z)$ is the component of the electric field which is tangent to the planes $x = 0, z \geq 0$ and $x = a, z \geq 0$, we have

$$\begin{aligned} 0 = E_y^{inc}(0, z) + \frac{i}{4}\int_0^\infty &\{I_0(z')H_0^{(1)}[k\,|\,z - z'\,|] \\ &+ I_1(z')H_0^{(1)}[k(a^2 + (z - z')^2)^{1/2}]\}\, dz', \end{aligned} \tag{1.2a}$$

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[2]Paley and Wiener, *The Fourier transform in the complex domain*, Am. Math. Society Colloquium Publication, 1934, ch. IV. Actually, the integral equations we are required to solve are singular cases of the Wiener-Hopf theory but they are still susceptible to Fourier techniques.

and

$$0 = E_y^{inc}(a, z) + \frac{i}{4} \int_0^\infty \{I_0(z')H_0^{(1)}[k(a^2 + (z - z')^2)^{1/2}] + I_1(z')H_0^{(1)}[k \mid z - z' \mid]\} \, dz' \tag{1.2b}$$

for $z \geq 0$.

We can simplify these last equations by performing the arithmetical operations of addition and subtraction. Upon adding, we get

$$0 = E_y^{inc}(0, z) + E_y^{inc}(a, z) + \frac{i}{4} \int_0^\infty J_0(z')\{H_0^{(1)}[k \mid z - z' \mid] + H_0^{(1)}[k(a^2 + (z - z')^2)^{1/2}]\}, \tag{1.3a}$$

while subtraction gives us immediately

$$0 = E_y^{inc}(0, z) - E_y^{inc}(a, z) + \frac{i}{4} \int_0^\infty J_1(z')\{H_0^{(1)}[k \mid z - z' \mid] - H_0^{(1)}[k(a^2 + (z - z')^2)^{1/2}]\}. \tag{1.3b}$$

Here $J_0(z) = I_0(z) + I_1(z)$ and $J_1(z) = I_0(z) - I_1(z)$. In view of the z dependence of the kernels and the particular limits of the integrals in Eqs. (1.3a) and (1.3b), we have here two integral equations which may be solved rigorously with the Fourier transform in the complex domain. This implies, of course, that we seek solutions of appropriate growth, and the kernels possess the correct growth. We shall now show that such is indeed the case.

2. The Fourier transform solution of equations (1.3a) and (1.3b). Let us now write Eqs. (1.3a) and (1.3b) in a form which makes them amenable to Fourier transform methods. We define $E_y^{inc}(0, z)$ and $E_y^{inc}(a, z)$, $J_0(z)$ and $J_1(z)$ to be identically zero for $z < 0$. We further extend the Eqs. (1.3a) and (1.3b) for $z < 0$ to read

$$\phi_0(z) = \frac{i}{4} \int_0^\infty J_0(z')\{H_0^{(1)}[k \mid z - z' \mid] + H_0^{(1)}[k(a^2 + (z - z')^2)^{1/2}]\} \, dz', \tag{2.1a}$$

$$\phi_1(z) = \frac{i}{4} \int_0^\infty J_1(z')\{H_0^{(1)}[k \mid z - z' \mid] - H_0^{(1)}[k(a^2 + (z - z')^2)^{1/2}]\} \, dz', \tag{2.1b}$$

where $\phi_0(z)$ and $\phi_1(z)$ are defined to be identically zero for $z > 0$. Upon noting our assumptions on $J_0(z)$, $J_1(z)$, $E_y^{inc}(0, z)$, $E_y^{inc}(a, z)$, $\phi_0(z)$ and $\phi_1(z)$ we have for all z

$$\phi_0(z) = E_y^{inc}(0, z) + E_y^{inc}(a, z) + \frac{i}{4} \int_{-\infty}^\infty J_0(z')\{H_0^{(1)}[k \mid z - z' \mid] + H_0^{(1)}[k(a^2 + (z - z')^2)^{1/2}]\} \, dz', \tag{2.2a}$$

$$\phi_1(z) = E_y^{inc}(0, z) - E_y^{inc}(a, z) + \frac{i}{4} \int_{-\infty}^\infty J_1(z')\{H_0^{(1)}[k \mid z - z' \mid] - H_0^{(1)}[k(a^2 + (z - z')^2)^{1/2}]\} \, dz'. \tag{2.2b}$$

We assume as in CHI that k has a small positive imaginary part.

Some remarks on the growth of $J_0(z)$, $J_1(z)$, $\phi_0(z)$ and $\phi_1(z)$ as z becomes either positively or negatively infinite are now in order. With this information we can find the half planes of regularity of the Fourier transforms of the functions with which we have to work. It is to be noted, in light of the definitions we have imposed upon $\phi_0(z)$ and $\phi_1(z)$, that they are asymptotic to $e^{-ikz}/z^{1/2}$ for z large and negative. This asymptotic form may be seen directly from the Hankel function. Thus the Fourier transforms of $\phi_0(z)$ and $\phi_1(z)$ are

$$\Phi_0(w) = \int_{-\infty}^{0} e^{-iwz}\phi_0(z)\,dz \qquad \text{and} \qquad \Phi_1(w) = \int_{-\infty}^{0} e^{-iwz}\phi_1(z)\,dz$$

and $\Phi_0(w)$ and $\Phi_1(w)$ are regular in an upper half plane $\mathfrak{Im}\,w > -\mathfrak{Im}\,k$.

The transforms of the Hankel functions $i/4\ H_0^{(1)}[k\mid z\mid]$ and $i/4\ H_0^{(1)}[k(a^2 + z^2])^{1/2}$ are well known. For example[3]

$$\frac{i}{4}\int_{-\infty}^{\infty} H_0^{(1)}[k(a^2 + z^2)^{1/2}]e^{-iwz}\,dz = \frac{i}{2}(k^2 - w^2)^{-1/2}\exp\,[i\mid a\mid(k^2 - w^2)^{1/2}]$$

and is regular in the strip $-\mathfrak{Im}\,k < \mathfrak{Im}\,w < \mathfrak{Im}\,k$. Furthermore the transforms of $E_y^{inc}(0, z)$ and $E_y^{inc}(a, z)$ are readily calculated since they have been annihilated for $z < 0$. Hence we have

$$\int_0^{\infty} e^{-iwz}E_y^{inc}(0, z)\,dz = \frac{1}{i(w - k\cos\theta)}$$

and

$$\int_0^{\infty} e^{-iwz}E_y^{inc}(a, z)\,dz = \frac{e^{iak\sin\theta}}{i(w - k\cos\theta)}$$

and these last two transforms are regular in the lower half plane $\mathfrak{Im}\,w < \mathfrak{Im}\,k\cos\theta$. We observe, that thus far the transforms of $\phi_0(z)$, $\phi_1(z)$, the Hankel functions, $E_y^{inc}(0, z)$ and $E_y^{inc}(a, z)$ are regular in a common strip $-\mathfrak{Im}\,k < \mathfrak{Im}\,w < \mathfrak{Im}\,k\cos\theta$.

We still have to discuss the growth properties of the surface current densities $I_0(z)$ and $I_1(z)$ for $z \gg 0$. Their dominant parts for $z \gg 0$ are terms of the type $e^{i\kappa z}$. All other terms in the asymptotic forms of $I_0(z)$ and $I_1(z)$ approach zero more rapidly than these imaginary exponentials. Now a term of the type $e^{i\kappa z}$ has the Fourier transform

$$\int_0^{\infty} e^{i\kappa z - iwz}\,dz$$

which is regular in some lower half plane bounded by a small but positive ordinate. It now follows that the Fourier transforms of $\phi_0(z)$, $\phi_1(z)J_0(z)$, $J_1(z)E_y^{inc}(a, z)$, $E_y^{inc}(0, z)$ and the Hankel functions are regular in the strip $-\mathfrak{Im}\,k < \mathfrak{Im}\,w < \mathfrak{Im}\,\kappa$ (or $\mathfrak{Im}\,k\cos\theta$). We are thus permitted to apply the Fourier transform to Eqs. (2.2a) and (2.2b) to get

$$\Phi_0(w) = \frac{(1 + e^{iak\sin\theta})}{i(w - k\cos\theta)} + \frac{i}{2}\frac{(1 + e^{ia(k^2-w^2)^{1/2}})}{(k^2 - w^2)^{1/2}}H_0(w), \tag{2.4a}$$

$$\Phi_1(w) = \frac{(1 - e^{iak\sin\theta})}{i(w - k\cos\theta)} + \frac{i}{2}\frac{(1 - e^{ia(k^2-w^2)^{1/2}})}{(k^2 - w^2)^{1/2}}H_1(w), \tag{2.4b}$$

[3]The branch of $(k^2 - w^2)^{1/2}$ is equal to k for $w = 0$.

where

$$H_0(w) = \int_0^\infty e^{-iwz} J_0(z)\, dz, \qquad H_1(w) = \int_0^\infty e^{-iwz} J_1(z)\, dz.$$

Equations (2.4a) and (2.4b) are now to be decomposed into two sets of equations, one of which will be analytic in the upper half plane $\mathfrak{Im}\, w > -\mathfrak{Im}\, k$ while the other of them will be regular in the lower half plane $\mathfrak{Im}\, w < \mathfrak{Im}\, k$ or ($\mathfrak{Im}\, k \cos \theta$). Let us first turn to Eq. (2.4a). The factor

$$(k^2 - w^2)^{-1/2}\{1 + \exp [ia(k^2 - w^2)^{1/2}]\}$$

may be written as

$$2(k^2 - w^2)^{-1/2} \exp \left[i \frac{a}{2} (k^2 - w^2)^{1/2} \right] \cos \left[\frac{a}{2} (k^2 - w^2)^{1/2} \right] = \frac{K_-(w)}{K_+(w)} = K(w).$$

Without indicating the precise form of $K_-(w)$ and $K_+(w)$ which we assume to be regular in the appropriate lower and upper half planes, we may proceed to the required decomposition of Eq. (2.4a). We have

$$\Phi_0(w)K_+(w) - \frac{(1 + e^{iak\sin\theta})[K_+(w) - K_+(k \cos \theta)]}{i(w - k \cos \theta)}$$
$$= \frac{(1 + e^{iak\sin\theta})K_+(k \cos \theta)}{i(w - k \cos \theta)} + \frac{i}{2} K_-(w)H_0(w). \tag{2.5a}$$

The left side of Eq. (2.5a) is regular in the upper half plane $\mathfrak{Im}\, w > -\mathfrak{Im}\, k$, while the right side is regular in the lower half plane $\mathfrak{Im}\, w < \mathfrak{Im}\, \kappa$ (or $\mathfrak{Im}\, k \cos \theta$) and both sides are regular in a common strip. It follows then that each side is equal to an integral function $\epsilon_0(w)$, i.e.

$$\Phi_0(w)K_+(w) - \frac{(1 + e^{iak\sin\theta})[K_+(w) - K_+(k \cos \theta)]}{i(w - k \cos \theta)} = \epsilon_0(w), \tag{2.6a}$$

$$\frac{(1 + e^{iak\sin\theta})K_+(k \cos \theta)}{i(w - k \cos \theta)} + \frac{i}{2} K_-(w)H_0(w) = \epsilon_0(w). \tag{2.6b}$$

In a similar fashion, we may decompose Eq. (2.4b). Let

$$(k^2 - w^2)^{-1/2}\{1 - \exp [ia(k^2 - w^2)^{1/2}]\} = \frac{L_-(w)}{L_+(w)} = L(w),$$

where $L_-(w)$ and $L_+(w)$ are regular in the appropriate lower and upper half planes. We have upon repeating the argument for separation

$$\Phi_1(w)L_+(w) - \frac{(1 - e^{iak\sin\theta})[L_+(w) - L_+(k \cos \theta)]}{i(w - k \cos \theta)} = \epsilon_1(w), \tag{2.7a}$$

$$\frac{i}{2} L_-(w)H_1(w) + \frac{(1 - e^{iak\sin\theta})L_+(k \cos \theta)}{i(w - k \cos \theta)} = \epsilon_1(w), \tag{2.7b}$$

where $\epsilon_1(w)$ is an integral function.

We are now compelled to indicate the precise forms of $K_-(w)$, $K_+(w)$, $L_-(w)$ and $L_+(w)$ if we are to evaluate the integral functions $\epsilon_0(w)$ and $\epsilon_1(w)$ and thereby find $H_0(w)$ and $H_1(w)$. In much the same manner which was indicated in CHI one finds that

$$K_-(w) = \frac{a^2}{\pi^2}\frac{(w-\kappa)}{(k-w)^{1/2}}\exp\left[\frac{ia}{\pi}(k^2-w^2)^{1/2}\arctan\left(\frac{k+w}{k-w}\right)^{1/2}\right.$$
$$\left.+\chi_0(w)\right]\prod_{n=1}^{\infty}\left[\left(1-\frac{(ak)^2}{\pi^2(2n+1)^2}\right)^{1/2}+\frac{iaw}{\pi(2n+1)}\right]e^{-iaw/\pi(2n+1)}$$

is regular in the appropriate lower half plane.[4] $\chi_0(w)$ is an integral function which has been introduced into the product decomposition of $K(w)$ and is to be chosen such that $K_-(w)$ is of algebraic growth for $|w| \to \infty$ and $\mathfrak{Im}\,w$ in the correct lower half plane. Similarly

$$\frac{1}{K_+(w)} = \frac{2(w+\kappa)}{(k+w)^{1/2}}\exp\left[\frac{ia}{\pi}(k^2-w^2)^{1/2}\arctan\left(\frac{k-w}{k+w}\right)^{1/2}\right.$$
$$\left.-\chi_0(w)\right]\prod_{n=1}^{\infty}\left[\left(1-\frac{(ak)^2}{\pi^2(2n+1)^2}\right)^{1/2}-\frac{iaw}{\pi(2n+1)}\right]e^{iaw/\pi(2n+1)}.$$

In order to determine $\chi_0(w)$, we simply calculate the asymptotic form of $K_-(w)$ and $K_+(w)$ as $|w| \to \infty$, $\mathfrak{Im}\,w$ in the appropriate half plane and choose $\chi_0(w)$ so that $K_-(w)$ and $K_+(w)$ will be of algebraic growth.[5] Let us first study $K_-(w)$. If we observe that the parameter ak may be neglected in the infinite product as $|w| \to \infty$, $\mathfrak{Im}\,w < \mathfrak{Im}\,k$, we have that $K_-(w)$ is asymptotic to

$$\begin{aligned} &w^{1/2}\exp\left[\frac{iaw}{2\pi}\log\left(-\frac{2w}{k}\right)+\chi_0(w)\right]\prod_{n=1}^{\infty}\left[1+\frac{iaw}{\pi(2n+1)}\right]e^{-iaw/\pi(2n+1)} \\ &\quad = cw^{-1/2}\frac{\Gamma(iaw/2\pi)}{\Gamma(iaw/\pi)}\exp\left[\frac{iaw}{\pi}\left(1-\frac{\gamma}{2}+\frac{1}{2}\log-\frac{2w}{k}\right)+\chi_0(w)\right] \end{aligned} \tag{2.8}$$

where c is a constant whose precise form does not interest us and γ is the Euler-Mascheroni constant. We may now apply the Stirling formula to (2.8) to obtain that $K_-(w)$ and $K_+(w)$ are of algebraic growth for $|w| \to \infty$, $\mathfrak{Im}\,w$ in the appropriate half planes, if

$$\chi_0(w) = \frac{iaw}{2\pi}\left[-3+\gamma-\log\frac{\pi}{ak}-\frac{i\pi}{2}\right].$$

With $\chi_0(w)$ so chosen, $K_-(w)$ is asymptotic to $w^{-1/2}$ for $|w| \to \infty$ and $\mathfrak{Im}\,w$ in the appropriate lower half plane, while $K_+(w)$ is asymptotic to $w^{1/2}$ for $|w| \to \infty$ and $\mathfrak{Im}\,w$ in the upper half plane $\mathfrak{Im}\,w > -\mathfrak{Im}\,k$.

We are now in a position to determine the integral function $\epsilon_0(w)$. Let us note in Eq. (2.6b) that $H_0(w)$ is the unilateral Fourier transform of $I_0(z) + I_1(z) = J_0(z)$. As such, since $J_0(z)$ has appropriate growth for z large and positive, and since it is integrable over any finite interval of z including the origin, $H_0(w)$ possesses the property that

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[4]Henceforth principal determinations of inverse trigonometric functions and logarithms are understood.

[5]J. S. Schwinger, *Theory of guided waves*, Radiation Laboratory publication, forthcoming.

it approaches zero for $|w| \to \infty$, $\mathfrak{Im}\, w < \mathfrak{Im}\, \kappa$ or $\mathfrak{Im}\, k \cos\theta$. If we let $|w| \to \infty$, w in the correct lower half plane we see from Eq. (2.6b) that $\epsilon_0(w) = O(w^{-\alpha-1/2})$, $\alpha > 0$. On the other hand $\Phi_0(w)$ approaches zero for $|w| \to \infty$ $\mathfrak{Im}\, w > -\mathfrak{Im}\, k$ because $\Phi_0(w)$ is the unilateral Fourier transform of a function defined for negative z, is integrable over any finite negative range of z including the origin and possesses appropriate growth for z large and negative. Hence for $|w| \to \infty$, $\mathfrak{Im}\, w > -\mathfrak{Im}\, k$, we see from Eq. (2.6a) that $\epsilon_0(w) = O(w^{1/2-\beta})$, $\beta > 0$. It follows, therefore, by a theorem of Liouville, that $\epsilon_0(w)$ is a polynomial of degree less than minus one half, and hence identically zero. We have finally

$$H_0(w) = \frac{2(1 + e^{iak\sin\theta})K_+(k\cos\theta)}{(w - k\cos\theta)K_-(w)}, \tag{2.9}$$

the Fourier transform of $J_0(z)$.

We can obtain some information regarding $J_0(z)$ for $z \to 0^+$ from $H_0(w)$ as $|w| \to \infty$, $\mathfrak{Im}\, w < \mathfrak{Im}\, \kappa$ or $\mathfrak{Im}\, k \cos\theta$. For now we have that

$$H_0(w) = O(w^{-1/2})$$

and this tells us immediately that

$$J_0(w) = O(z^{-1/2})$$

for $z \to 0^+$. This verifies the integrability of $J_0(z)$ for finite and positive z.

We now turn to the determination of $H_1(w)$ and for this we consider Eq. (2.4b). We note that

$$\frac{L_-(w)}{L_+(w)} = -2i(k^2 - w^2)^{-1/2} \exp\left[\frac{ia}{2}(k^2 - w^2)^{1/2}\right] \sin\left[\frac{a}{2}(k^2 - w^2)^{1/2}\right]$$

where now

$$L_-(w) = -ia \exp\left[\frac{ia}{\pi}(k^2 - w^2)^{1/2} \operatorname{arc\,tan}\left(\frac{k+w}{k-w}\right)^{1/2} + \chi_1(w)\right] \prod_{n=1}^{\infty}\left[\left(1 - \left(\frac{ak}{2\pi n}\right)^2\right)^{1/2} + \frac{iaw}{2\pi n}\right] e^{-iaw/2\pi n}$$

and

$$\frac{1}{L_+(w)} = \exp\left[\frac{ia}{\pi}(k^2 - w^2)^{1/2} \operatorname{arc\,tan}\left(\frac{k-w}{k+w}\right)^{1/2} - \chi_1(w)\right] \prod_{n=1}^{\infty}\left[\left(1 - \left(\frac{ak}{2\pi n}\right)^2\right)^{1/2} - \frac{iaw}{2\pi n}\right] e^{iaw/2\pi n}.$$

Here we have the $L_-(w)$ which is regular in the lower half plane $\mathfrak{Im}\, w < \mathfrak{Im}\, k$ and the $L_+(w)$ which is regular in the upper half plane $\mathfrak{Im}\, w > -\mathfrak{Im}\, k$. Again we choose $\chi_1(w)$, an integral function, such that $L_-(w)$ and $L_+(w)$ will behave algebraically as $|w| \to \infty$ and $\mathfrak{Im}\, w$ in either of the appropriate half planes just described. We proceed as we did above to determine $\chi_1(w)$. For $|w| \to \infty$, $\mathfrak{Im}\, w < \mathfrak{Im}\, k$, $L_-(w)$ is asymptotic to

$$\exp\left[\chi_1(w) + \frac{iaw}{2\pi}\log - \frac{2w}{k}\right]\prod_{n=1}^{\infty}\left[1 + \frac{iaw}{2\pi n}\right]e^{-iaw/2\pi n}$$

$$= \frac{2\pi \exp\left[\chi_1(w) - iaw\gamma/2\pi + \{iaw/2\pi\}\log(-2w/k)\right]}{iaw\Gamma(iaw/2\pi)}.$$

Upon applying Stirling's expansion theorem we obtain immediately

$$\frac{2\pi}{iaw}\exp\left[\chi_1(w) + \{iaw/2\pi\}\{1 - \gamma + \log(-2w/k)\}\right]\left(\frac{iaw}{2\pi}\right)^{1/2-iaw/2\pi}$$

Hence, if we choose

$$\chi_1(w) = -\frac{iaw}{2\pi}\left[1 - \gamma + \log\frac{4\pi}{ak} + \frac{i\pi}{2}\right]$$

$L_-(w)$ will have algebraic growth in the lower half plane $\Im mw < \Im mk$ for $|w| \to \infty$ and will be asymptotic to $w^{-1/2}$. If we repeat the argument in the upper half plane, $\Im mw > -\Im mk$ for the term $L_+(w)$ we find that the same $\chi_1(w)$ will render $L_+(w)$ algebraic in growth for $|w| \to \infty$ and now $L_+(w)$ will be asymptotic to $w^{1/2}$ for $w \to \infty$. Finally, reasoning as we did for Eqs. (2.6a) and (2.6b) we find that $\epsilon_1(w)$ is $O(w^{\alpha_1})$, $\alpha_1 < -\frac{1}{2}$ for $|w| \to \infty$, $\Im mw < \Im mk$ and is $O(w^{\beta_1})$, $\beta_1 < \frac{1}{2}$ for $|w| \to \infty$, $\Im mw > -\Im mk$. Applying Liouville's theorem once again, we find that $\epsilon_1(w)$ is identically zero. Thus we now have

$$H_1(w) = \frac{2(1 - e^{iak\sin\theta})L_+(k\cos\theta)}{(w - k\cos\theta)L_-(w)}$$

the Fourier transform of $I_0(z) - I_1(z) = J_1(z)$.

From $H_0(w)$ and $H_1(w)$ we can obtain the Fourier transforms of $I_0(z)$ and $I_1(z)$ by a simple addition and subtraction. We can also see how $J_1(z)$ behaves for $z \to 0^+$. Since $H_1(w)$ is now $O(w^{-1/2})$ for $|w| \to \infty$, $\Im mw < \Im mk$, $J_1(z) = O(z^{-1/2})$ for $z \to 0^+$ and this verifies the integrability of $J_1(z)$ for finite and positive z. In closing, we note that the precise forms of $I_0(z)$ and $I_1(z)$ are of no interest to us since we are only interested in the far fields.

3. The calculation of the far fields. In order to calculate the far fields we first express Eq. (1.3) by a Fourier integral representation. We have

$$E_y(x, z) = \exp\left[ik(x\sin\theta + z\cos\theta)\right]$$

$$+ \frac{i}{8\pi}\int_C e^{iwz}(k^2 - w^2)^{-1/2}\left[\{H_0(w) + H_1(w)\}e^{i|x|(k^2-w^2)^{1/2}}\right.$$

$$\left. + \{H_0(w) - H_1(w)\}e^{i|x-a|(k^2-w^2)^{1/2}}\right] dw$$

where C is a path of integration drawn within the strip of regularity of all the Fourier transforms which appear in the above integral. The path is closed either above or below depending upon whether $z > 0$ or $z < 0$. Care must be taken in closing the path so that it does not intersect the branch cuts which are introduced due to the presence of the branch points k and $-k$. The dominant terms arise from the residues due to the two poles $k\cos\theta$ and κ. Furthermore, since contributions from other poles or branch points

give rise to terms which are small compared to the terms arising from the poles $k \cos \theta$ and κ for $|z| \to \infty$, we need only calculate these dominant effects, at least insofar as we are concerned with the far field.[6]

There are four separate regions of interest (a) $z < 0$, $-\infty < x < \infty$ (b) $z > 0$, $x \geq a$ (c) $z \geq 0$, $x \leq 0$ and (d) $z \geq 0$, $0 \leq x \leq a$. Let us consider region (a). For $z \leq 0$, $E_y(x, y)$ is asymptotic to $\exp [ik(z \cos \theta + x \sin \theta)]$, as it should. For region (b), $z \gg 0$, $x > a$, $E_y(x, z)$ has no term comparable in magnitude with the plane wave term. Thus region (b) is the region of the geometrical shadow. For region (c) $z \gg 0$, $x < 0$, $E_y(x, z)$ is asymptotic to

$$2i \exp [ikz \cos \theta] \sin [kx \sin \theta].$$

Thus for $x < 0$, $z \gg 0$, the lower plate acts as a perfect reflector.

Region (d) is the interesting one. We now have a means of finding the amplitude of the transmitted wave guide mode. We know that in this region $E_y(x, z)$ is asymptotic to $e^{i\kappa z} \sin \pi x/a$ for $z \gg 0$, $0 \leq x \leq a$. On the other hand when we evaluate the integral and take out its dominant terms, we find that we are left with

$$-\frac{ai}{\pi}(1 + e^{iak\sin\theta}) \frac{K_+(k \cos \theta)}{\kappa - k \cos \theta} \lim_{w \to \kappa} \frac{w - \kappa}{K_-(w)} e^{i\kappa z} \sin \pi x/a.$$

The amplitude of the transmitted wave is the coefficient of the factor $e^{i\kappa z} \sin \pi x/a$. It may be simplified if we now take k to be real. In the first place

$$\lim_{w \to \kappa} \frac{w - \kappa}{K_-(w)} = 2[\pi^3(k - \kappa)a^{-2}]^{1/2} \exp \left[i\Theta_1 - \chi_0(\kappa) - i \arctan \left(\frac{k + \kappa}{k - \kappa} \right)^{1/2} \right]$$

where

$$\Theta_1 = -\sum_{n=1}^{\infty} \left[\arcsin \frac{\kappa a}{\pi[(2n + 1)^2 - 1]^{1/2}} - \frac{\kappa a}{\pi(2n + 1)} \right]$$

while

$$K_+(k \cos \theta) = \frac{a[k(1 + \cos \theta)(k^2 \cos^2 \theta - \kappa^2)]^{1/2} \exp [i\Theta_2 + \chi_0(k \cos \theta) - (iak\theta \sin \theta)/2\pi]}{2\pi(k \cos \theta + \kappa)[\cos \{(ak \sin \theta)/2\}]^{1/2}}$$

and

$$\Theta_2 = \sum_{n=1}^{\infty} \left[\arcsin \frac{ak \cos \theta}{[\pi^2(2n + 1)^2 - (ak \sin \theta)^2]^{1/2}} - \frac{ak \cos \theta}{\pi(2n + 1)} \right].$$

Thus, the transmission coefficient is

$$2\left[\frac{k\pi(1 + \cos \theta) \cos (ak/2 \sin \theta)}{a^2(k^2 \cos^2 \theta - \kappa^2)(k + \kappa)} \right]^{1/2} e^{i\Psi + a(k\cos\theta - \kappa)/4}, \qquad 0 \leq \theta \leq \pi$$

[6]E. T. Copson, Oxford Quart. Math. **17**, 19-34 (1946). There is a detailed discussion in this paper on the choice of a path similar to C.

where

$$\Psi = \Theta_1 + \Theta_2 + \arctan \left(\frac{k - \kappa}{k + \kappa}\right)^{1/2} + \frac{ak}{2}\left(1 - \frac{\theta}{\pi}\right)\sin\theta$$

$$+ \frac{a(k\cos\theta - \kappa)}{2\pi}[\gamma - 3 - \log(\pi/ak)].$$

The square of the absolute magnitude of the amplitude of the transmission coefficient is proportional to the power gain, as a direct consequence of the Lorentz reciprocity theorem. Thus insofar as the angular variation is concerned, the radiation pattern is

$$\frac{(1 + \cos\theta)\cos(ak/2\sin\theta)e^{(ak\cos\theta)/2}}{(k^2\cos^2\theta - \kappa^2)}.$$

We have thus found how the parallel plates act as a receiving antenna. It is to be noted that the radiation pattern arises from the excitation of the parallel plates for $z \gg 0$, $0 \leq x \leq a$. The reciprocity theorem has enabled us to give a partial solution of the second part of the problem. The reflection coefficient which we have described earlier has yet to be calculated.

A PRACTICAL METHOD FOR SOLVING HILL'S EQUATION*

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1. Introduction. The differential equation known as Mathieu-Hill's equation can be written

$$y'' + J(x)y = 0, \tag{1}$$

where J is a periodic function of x. The period is usually taken equal to π for historical reasons. The first equation of this type was discovered by Mathieu in connection with the problem of vibrations within an elliptic boundary, when

$$J(x) = \eta + \gamma \cos 2x$$

has a period π.

Floquet proved that the general solution of Eq. (1) can be written

$$y = D_1 e^{\mu x}\Phi(x) + D_2 e^{-\mu x}\Phi(-x), \tag{2}$$

where Φ is a periodic function, with the same period π as $J(x)$. This general solution contains two terms with the exponents $\pm\mu$. Floquet's theorem can be expressed in a slightly different way. Let us consider the term with $+\mu$:

$$f(x) = e^{\mu x}\Phi(x). \tag{3}$$

The condition on $f(x)$ is

$$f(x + n\pi) = e^{\mu n \pi} f(x) = \xi^n f(x)$$

with

$$\xi = e^{\mu\pi}.$$

The general solution is

$$y = D_1 f(x) + D_2 f(-x). \tag{4}$$

We shall look for a solution in an interval of length π and use condition (3) to extend the solution from $-\infty$ to $+\infty$. This means that we shall have to meet some boundary conditions in order to match the solutions in two consecutive intervals. These matching conditions will be essential in fixing the value of μ.

The method developed in this paper is based upon these general considerations and shall be explained more completely in Sec. 2. It differs completely from the classical method, as found in most textbooks.[1] The standard procedure is to expand J in Fourier series

$$J = \sum_n \theta_n e^{i2nx} \tag{5}$$

and to look for a similar expansion for the unknown periodic function Φ

$$\Phi = \sum_n b_n e^{i2nx}. \tag{6}$$

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[1]Whittaker and Watson, *Modern analysis*, Cambridge University Press, 4th edition, 1927, p. 414.

Equation (1) results in an infinite system of simultaneous linear homogeneous equations for the unknown b_n s. A non-trivial solution can only be obtained if the corresponding infinite determinant is zero. This last condition is used to determine the value of the exponent μ.

Whittaker was able to discuss this condition in the case when the series of the coefficients θ_n in expansion (5) is absolutely convergent, and obtained a formula

$$\sin^2\left(\frac{\pi}{2}\, i\mu\right) = -\sinh^2\left(\frac{\pi}{2}\,\mu\right) = \Delta_1(0)\sin^2\left(\frac{\pi}{2}\,\theta_0^{1/2}\right), \tag{7}$$

where $\Delta_1(0)$ is another infinite determinant with the following coefficients

$$\Delta_1(0) = |\, B_{mp}\,|, \qquad B_{mm} = 1,$$
$$B_{mp} = \frac{\theta_{m-p}}{\theta_0 - 4m^2}, \qquad (m \neq p) \tag{8}$$

This result is not very encouraging. First, the condition of absolute convergence for the series of the coefficients of θ_n is a very restrictive one. Second, we still have to compute an infinite determinant, and the computation proves very difficult unless the θ_n terms decrease very rapidly when n increases.

This infinite determinant takes on infinite value whenever

$$\theta_0 = 2n \qquad n \text{ integer} \tag{9}$$

These θ_0 values correspond to double poles of the determinant, since both rows $m = \pm n$ obtain infinite terms. These double poles are canceled out in formula (7) by the double zeros of $\sin^2(\pi\theta_0^{1/2}/2)$ and do not correspond to any singular values of μ.

Altogether, the method of Fourier expansion is not very practical, and leads to complicated computations.

The method presented in this paper does not involve the restrictions of the classical method, and leads to a practical solution of Hill's equation, even in such exceptional cases as periodic functions $J(x)$ containing discontinuities or δ functions.

2. Principle of the method. Let us consider a differential equation

$$y'' + F(x)y = 0 \tag{10}$$

with a given function $F(x)$. We may find two independent solutions u and v and obtain the general solution

$$y = Au + Bv \tag{11}$$

containing two constants A and B. We obviously have

$$uv'' = vu'' = Fuv,$$

hence

$$uv' - vu' = C,$$

and a suitable normalization of u and v is used to make the constant C unity:

$$uv' - vu' = 1 \tag{12}$$

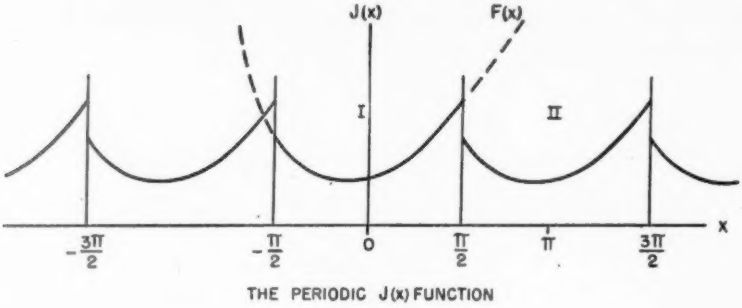

FIG. 1

We now want to discuss Hill's equation

$$y'' + J(x)y = 0 \tag{13}$$

with a periodic function J of period π defined in the following way (see Fig. 1):

$$J = \begin{cases} F(x) & -\pi/2 < x < \pi/2, \\ F(x - \pi) & \pi/2 < x < 3\pi/2, \\ \dots\dots\dots\dots\dots\dots & \\ F(x - n\pi) & n\pi - \pi/2 < x < n\pi + \pi/2. \end{cases} \tag{14}$$

Floquet's theorem assures us of the existence of two independent solutions y_1, y_2 characterized by the following properties (Eq. 5):

$$\begin{aligned} y_1(x + \pi) &= e^{\mu\pi}\, y_1(x) = \xi y_1(x), \\ y_2(x + \pi) &= e^{-\mu\pi} y_2(x) = \xi^{-1} y_2(x), \\ \xi &= e^{\mu\pi}. \end{aligned} \tag{15}$$

Let us consider y_1 and discuss a practical method for obtaining μ (or ξ). In the first interval $(-\pi/2 < x < \pi/2)$ the function y_1 may be represented by a formula (11), with a set of A and B coefficients. In the second interval $(\pi/2 < x < 3\pi/2)$ the coefficients will be ξA and ξB, according to (15). We must now write the continuity conditions for y_1 and y_1' across the border $\pi/2$:

$$\begin{aligned} Au_1 + Bv_1 &= \xi A u_2 + \xi B v_2, \\ Au_1' + Bv_1' &= \xi A u_2' + \xi B v_2', \end{aligned} \tag{16}$$

$$u_1 = u(\pi/2), \qquad u_2 = u(-\pi/2), \qquad v_1 = v(\pi/2), \qquad v_2 = v(-\pi/2).$$

We have obtained a set of simultaneous equations for the two unknowns A and B. This can be solved only when the determinant is zero:

$$D = \begin{vmatrix} u_1 - \xi u_2 & v_1 - \xi v_2 \\ u_1' - \xi u_2' & v_1' - \xi v_2' \end{vmatrix} \tag{17}$$

$$= \xi^2 + \xi(u_1'v_2 + u_2'v_1 - u_1v_2' - u_2v_1') + 1 = 0,$$

where we have used Eq. (12) at both points $\pm\pi/2$. Equation (17) in ξ fixes the Floquet coefficients ξ and ξ^{-1}. The product of the two roots is unity and their sum is given by

$$2 \cosh \mu\pi = \xi + \xi^{-1} = -u_1'v_2 - u_2'v_1 + u_1v_2' + u_2v_1'; \tag{18}$$

hence

$$4 \sinh^2 \left(\mu \frac{\pi}{2}\right) = 2 \cosh \mu\pi - 2 = -(u_1 - u_2)(v_1' - v_2') + (u_1' - u_2')(v_1 - v_2) \tag{19}$$

with the help of Eq. (12). Once the μ exponent is obtained, the coefficients A and B result from (16) and the solution of Hill's equation (13) is achieved.

A very important case is obtained when

$$F(x), \qquad J(x) = \text{even}. \tag{20}$$

One may choose correspondingly for u, v an even and an odd function:

$$\begin{aligned} u(x) &= u(-x), \qquad u_2 = u_1, \qquad u_2' = -u_1', \\ v(x) &= -v(-x), \qquad v_2 = -v_1, \qquad v_2' = v_1'. \end{aligned} \tag{21}$$

Eq. (18) then reads

$$\cosh \mu\pi = u_1v_1' + u_1'v_1 \tag{22}$$

from which we obtain

$$\sinh^2 (\mu\pi/2) = u_1'v_1, \tag{23}$$

a formula that will be found useful for a comparison with Whittaker's theory of Hill's equation.

The whole method is a generalization of the discussion that was previously given for the special example of a rectangular $J(x)$ function.[2] One advantage of the method is that it works with periodic $J(x)$ functions exhibiting a finite number of discontinuities.

3. Some special examples. The case of a rectangular $J(x)$ function can be easily investigated along these lines, and the results agree completely with those of a previous discussion[2] where a slightly different method was followed.

Let us consider the case of a parabolic function

$$F = a - b^2x^2. \tag{24}$$

[2]L. Brillouin, *Wave propagation in periodic structures*, McGraw-Hill, New York, 1946, Ch. VIII, pp. 180-186, and Ch. IX, pp. 218-226.

We may find the solutions as power series expansions

$$\begin{aligned} u &= 1 + u_2x^2 + \cdots\cdot u_{2n}x^{2n} \cdots, \\ v &= x + v_3x^3 + \cdots\cdot v_{2n+1}x^{2n+1} \cdots. \end{aligned} \tag{25}$$

Substituting in Eq. (10) we obtain the recurrence formulas

$$\begin{aligned} 2u_2 + a = 0, \qquad (2n + 2)(2n + 1)u_{2n+2} + au_{2n} - b^2u_{2n-2} = 0, \\ 6v_3 + a = 0, \qquad (2n + 3)(2n + 2)v_{2n+3} + av_{2n+1} - b^2v_{2n-1} = 0, \end{aligned} \tag{26}$$

and the u, v functions satisfy the normalizing condition (12). The following special cases may be of interest

$$\begin{aligned} a = -b, \qquad & u = e^{bx^2/2}, \\ & v = e^{bx^2/2}I(x) \qquad \text{with } I(x) = \int_0^x e^{-bx^2}\,dx, \end{aligned} \tag{27}$$

$$\begin{aligned} a = -3b, \qquad & u = e^{-bx^2/2} + 2bx\,e^{bx^2/2}I(x), \\ & v = xe^{bx^2/2}, \end{aligned} \tag{28}$$

as may be checked by direct computation.

In all these cases, there is no discontinuity of the function F on the limits $\pm\pi/2$ of the interval, and the curve on Fig. 1 is a continuous curve with a discontinuous derivative at $\pm\pi/2$.

The corresponding Hill's problem is immediately solved with the help of Equation (22) or (23). For instance, the cases indicated above under (27) and (28) yield:

$$a = -b, \qquad \sinh^2\left(\mu\frac{\pi}{2}\right) = u_1'v_1 = b\frac{\pi}{2}e^{b\pi^2/4}I\left(\frac{\pi}{2}\right) \tag{29}$$

$$a = -3b, \qquad \sinh^2\left(\mu\frac{\pi}{2}\right) = b\frac{\pi^2}{4} + b\pi e^{b\pi^2/4}I\left(\frac{\pi}{2}\right) + b^2\frac{\pi^3}{4}e^{b\pi^2/4}I\left(\frac{\pi}{2}\right) \tag{30}$$

These results could not have been obtained by any other method of solution.

Another example can be solved with the help of Bessel functions, which satisfy the equation

$$z\frac{d}{dz}\left(z\frac{d}{dz}y\right) + (z^2 - n^2)y = 0, \tag{31}$$

$$y = AJ_n(z) + BJ_{-n}(z), \qquad n \text{ non integer} \tag{32}$$

Taking a new variable

$$x = \log z, \qquad z = e^x,$$

we obtain an equation of type (10):

$$\frac{d^2}{dx^2}y + (e^{2x} - n^2)y = 0. \tag{33}$$

Thus,

$$y = AJ_n(e^x) + BJ_{-n}(e^x) \tag{34}$$

and we have a solution corresponding to an unsymmetrical

$$F = e^{2x} - n^2 \tag{35}$$

for which our general formulas (18), (19) should be used in connection with the corresponding Hill's equation.

4. Solution with the B. W. K. method. The point of departure of our method is a solution of an equation of type (10). An approximate solution can be found with the B. W. K. procedure,[3] if $F(x)$ is exhibiting only small variations about a large average value. A more precise statement of the conditions involved will result from the following discussion. We rewrite Eq. (10) as follows:

$$y'' + G^2(x)y = 0, \qquad F(x) = G^2(x). \tag{36}$$

We now consider a function

$$y = G^{-1/2}\, e^{iS}, \qquad S = \int_0^x G\, dx, \tag{37}$$

which yields

$$\frac{y''}{y} = \frac{3}{4}\left(\frac{G'}{G}\right)^2 - \frac{1}{2}\frac{G''}{G} - G^2. \tag{38}$$

The function y represents an approximate solution of Eq. (36) if the first two terms in (38) are negligible in comparison to the last one. This is the case if

$$\frac{G'}{G^2} \sim \epsilon, \qquad \frac{G''}{G^3} \sim \epsilon^2, \qquad \epsilon^2 \ll 1, \tag{39}$$

and terms in ϵ^2 are neglected, while ϵ terms are retained. The second condition (39) is very restrictive, however, since it allows only for variations of G' of the order of ϵ^2. The function (37) becomes a *rigorous solution* of Eq. (36) when

$$\frac{3}{4}\left(\frac{G'}{G}\right)^2 - \frac{1}{2}\frac{G''}{G} = 0, \tag{40}$$

$$G = \frac{A}{(a + x)^2}. \tag{41}$$

Solutions u, v normalized in accordance with (12) are easily found:

$$u = (2iG)^{-1/2}e^{-iS}, \qquad v = (2iG)^{-1/2}e^{+iS} \tag{42}$$

An interesting example is shown in Fig. 2; it corresponds to the *even* function

$$G = \frac{A}{(a + |x|)^2} \tag{43}$$

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[3]The initials B. W. K. refer to the three authors L. Brillouin, Journ. de Phys. **7**, 353 (1926); G. Wentzel, Zts. f. Phys. **38**, 518 (1926); H. A. Kramers, Zts. f. Phys. **39**, 828 (1926).

For a complete discussion and more references, see E. C. Kemble, *The fundamental principles of quantum mechanics*, McGraw-Hill, 1937, Ch. III, p. 91-112.

with a discontinuity of the derivative at $x = 0$. Here,

$$-S(-x) = S(x) = \int_0^{x>0} G\,dx = -A\left(\frac{1}{a+x} - \frac{1}{a}\right) = \frac{Ax}{a(a+x)}.$$

For the *odd* solution v we simply take

$$\begin{aligned} v &= G^{-1/2}\sin S = \frac{a + |x|}{A^{1/2}}\sin S, \\ v' &= -\frac{1}{2}G^{-3/2}G'\sin S + G^{1/2}\cos S. \end{aligned} \tag{44}$$

Both v and v' are continuous at $x = 0$ since $G'(+0) = -G'(-0)$. The even solution u can be written as follows:

$$\begin{aligned} u &= G^{-1/2}[\cos S \pm K \sin S], \\ u' &= -\frac{1}{2}G^{-3/2}G'_{\pm}\,[\cos S \pm K\sin S] + G^{1/2}[-\sin S \pm K\cos S], \end{aligned} \tag{45}$$

where the $+$ sign must be taken for $0 < x < \pi/2$ and the minus sign for $-\pi/2 < x < 0$. The function u is continuous at the origin, and u' becomes continuous when it is made to vanish at the origin. Hence,

$$K = \frac{G'_+(0)}{2G^2(0)} = -\frac{a}{A}.$$

Finally, after some reductions, one finds

$$\begin{aligned} u &= \frac{a + |x|}{A^{1/2}}\left[\cos S \mp \frac{a}{A}\sin S\right], \\ u' &= \frac{1}{A^{1/2}}\left[\frac{|x|}{a + |x|}\cos S - \frac{A^2 + a(a + |x|)}{A(a + |x|)}\sin S\right]. \end{aligned} \tag{46}$$

It is easily verified that the coefficients have been chosen correctly so as to satisfy the normalizing conditions (12).

With these solutions u, v we may now compute the Floquet coefficient μ, with the help of Eq. (23). We find

$$\sinh^2\left(\mu\frac{\pi}{2}\right) = u_1'v_1 = \frac{|x|}{A}\sin S_1\cos S_1 - \sin^2 S_1\,\frac{A^2 + a(a + |x|)}{A^2},$$

where we must take

$$x = \pi/2, \qquad S_1 = \frac{A\pi/2}{a(a + \pi/2)}$$

$$\sinh^2\left(\mu\frac{\pi}{2}\right) = \frac{\pi}{2A}\sin S_1\cos S_1 - \frac{A^2 + a^2 + a\pi/2}{A^2}\sin^2 S_1\,. \tag{47}$$

A simple check can be made on this formula. Keeping the same A value, we may replace a by

$$a' = -a - \frac{\pi}{2},$$

thus interchanging the position of the angular maxima and minima of the curve (see Fig. 2). Our formula is not affected by this change.

5. Successive approximations. Starting from the solutions obtained in the preceding sections, we may develop a method of successive approximations. Let us consider Eq. (10) with a function F and assume that it can be approximately represented by a G^2 function of the type (41).

$$G = \frac{A}{(a + x)^2}, \qquad F = G^2 - \epsilon H(x), \qquad \epsilon \text{ small.} \tag{48}$$

If a single function G does not yield a good enough approximation over the whole interval $-\pi/2, +\pi/2$, it may be convenient to divide this interval into two or more partial intervals using different G functions, and to join solutions at the boundaries, as shown on Figs. 2 or 3.

Next we use an expansion

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots .$$

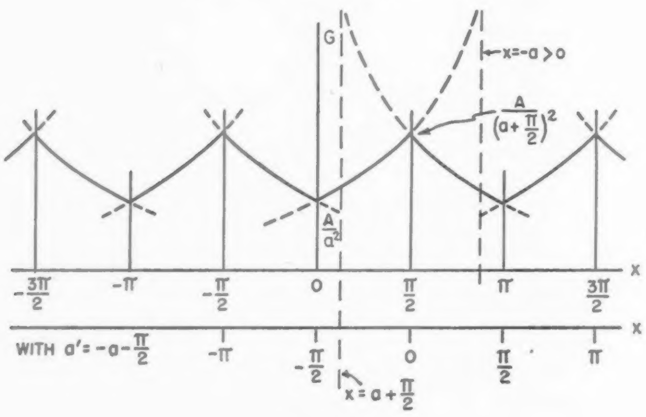

Fig. 2

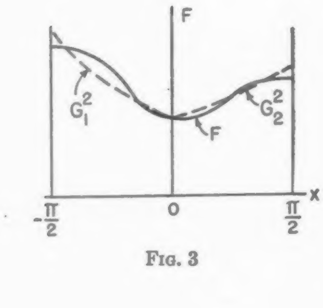

Fig. 3

Grouping terms with the same power of ϵ, we obtain

$$y_0'' + G^2 y_0 = 0, \tag{49a}$$

$$y_1'' + G^2 y_1 = Hy_0, \tag{49b}$$

$$y_2'' + G^2 y_2 = Hy_1. \tag{49c}$$

The solutions of (49a) are the u_0 and v_0 obtained in (42). Thus, the solution of (49b) reads

$$y_1 = u_0 \int_0^x Y_0 v_0' \, dx - v_0 \int_0^x Y_0 u_0' \, dx \qquad \text{with } Y_0 = \int_0^x Hy_0 \, dx \tag{50}$$

and the further approximations can be obtained in a similar way.

As an example, let us assume an even function F which can be approximated with the even G function (43). The zero order approximation is represented by u_0 and v_0 of equations (44), (45). We thus have,

$$U_0 = \int_0^x Hu_0 \, dx, \qquad V_0 = \int_0^x Hv_0 \, dx. \tag{51}$$

H being even, U_0 is odd and V_0 is even. Next, we obtain

$$\begin{aligned} u_1 &= u_0 \int_0^x U_0 v_0' \, dx - v_0 \int_0^x U_0 u_0' \, dx \qquad &\text{(even)}, \\ v_1 &= u_0 \int_0^x V_0 v_0' \, dx - v_0 \int_0^x V_0 u_0' \, dx \qquad &\text{(odd)}, \end{aligned} \tag{52}$$

and the first order solutions are

$$\begin{aligned} u &= u_0 + \epsilon u_1 + \cdots \qquad &\text{(even)}, \\ v &= v_0 + \epsilon v_1 + \cdots \qquad &\text{(odd)}. \end{aligned} \tag{53}$$

These functions being automatically normalized according to (12), as a direct check easily shows.

A solution of Eq. (10) can thus be obtained step by step to any desired degree of approximation, and may be used to the solution of Hill's equation as shown in Sec. 2.

The same method could be applied to any other known solution of Eq. (10). We used the G^2 functions in the zero order approximation, but any other known solution would do just as well.

6. Hill's equation containing delta function. Let us consider our fundamental Eq. (10) and assume the function F to be a delta function:

$$F = B\delta(x), \qquad \delta = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}, \qquad \int_{-\infty}^{+\infty} \delta(x) \, dx = 1 \tag{54}$$

Equation (10) can be readily integrated:

$$y'' = \begin{cases} 0, & (x \neq 0), \\ B\delta(x) y(x) & (x = 0), \end{cases}$$

Thus,

$$y' = \begin{cases} a, & (x < 0) \\ a + By(0), & (x > 0) \end{cases} \tag{55}$$

and we obtain our even u or odd v solutions:

$$\begin{aligned} u &= 1 + \frac{1}{2} B \mid x \mid & \text{(even)}, \\ v &= x & \text{(odd)}. \end{aligned} \tag{56}$$

These u, v functions are normalized according to (12). The solution of the corresponding Hill's equation is obtained from Eq. (23):

$$\sinh^2 \left(\mu \frac{\pi}{2}\right) = (u'v)_{x=\pi/2} = \left(\frac{1}{2} Bx\right)_{x=\pi/2} = \frac{\pi}{4} B \tag{57}$$

This result can be checked by different methods. The case of a rectangular J function was discussed in the author's book,[2] p. 181, assuming

$$J = \begin{cases} -\chi_1^2 & (-l_1 < x < 0) \\ -\chi_2^2 & (0 < x < l_2) \end{cases} \qquad (l_1 + l_2 = \pi) \tag{58}$$

The solution was given by equation (44.12), *loc. cit.* p. 181:

$$\cosh \mu\pi = \cosh \chi_1 l_1 \cosh \chi_2 l_2 + \frac{1}{2}\left(\frac{\chi_1}{\chi_2} + \frac{\chi_2}{\chi_1}\right) \sinh \chi_1 l_1 \sinh \chi_2 l_2 . \tag{59}$$

We now take

$$\chi_1 = 0, \qquad \chi_2 \to \infty, \qquad l_2 \to 0, \qquad l_2 \chi_2^2 = B,$$

and obtain

$$\cosh \mu\pi = 1 + \frac{1}{2} \chi_2 l_1 \chi_2 l_2 = 1 + \frac{B}{2} l_1 \tag{60}$$

where $l_1 = \pi$ when $l_2 = 0$. This checks with our previous result (57). Other types of J functions can be used and lead to similar results. Such a problem is completely outside the reach of the Fourier series expansion method.

7. Comparison between the present method and the classical one. We discussed in Sec. 1 the classical method of solution, and underlined its limitations. The Fourier expansion of the periodic function J must be such that the series of the Fourier coefficients be absolutely convergent. This rules out functions with discontinuities, whose Fourier coefficients decrease as slowly as $1/n$, but the classical method should apply to a continuous function J with discontinuous derivative. This is the case for the problem discussed in Sec. 4, Eq. (43):

$$F = G^2 = \frac{A^2}{(a + \mid x \mid)^4} \tag{61}$$

Within a period π this function oscillates between A^2/a^4 and $A^2/((a + \pi/2)^4)$. The corresponding periodic function J can be analysed in Fourier series and Whittaker's solution obtained. The question now is to compare solutions computed one way or the other.

The comparison is easier when the variation of the function is small; hence we assume

$$A = Ba^2, \qquad a \gg \frac{\pi}{2} \tag{62}$$

and compute expansions with respect to the small quantity

$$\epsilon = \frac{\pi}{2a} \ll 1.$$

Let us start with the solution obtained in Sec. 4. It contains the quantity S_1 (Eq. 47).

$$S_1 = \frac{A\pi/2}{a(a + \pi/2)} = \frac{B\pi/2}{1 + \pi/2a} = B\frac{\pi}{2}\left(1 - \frac{\pi}{2a} + \left(\frac{\pi}{2a}\right)^2 \cdots\right) \tag{63}$$

we shall keep terms up to $\pi/2a$ only, since the computation of second order terms proves rather cumbersome. The solution is given in Eq. (47), and contains

$$\sin S_1 = \sin(\pi B/2) - B(\pi^2/4a)\cos(\pi B/2) + \cdots,$$

$$\cos S_1 = \cos(\pi B/2) + B(\pi^2/4a)\sin(\pi B/2) + \cdots.$$

Using these expansions in Eq. (47), we note a first term in $(\pi/2A)\sin S_1 \cos S_1$ that can be dropped, since A is of the order of a^2. We thus are left with

$$\sinh^2\left(\mu\frac{\pi}{2}\right) = -\sin^2 S_1\left(1 + \frac{a(a + \pi/2)}{B^2a^4}\right)$$

and the term in parenthesis is again of second order. Thus,

$$\begin{aligned}\sinh^2\left(\mu\frac{\pi}{2}\right) &= -\sin^2 S_1 = -\sin^2(\pi B/2) + 2B\frac{\pi^2}{4a}\sin(\pi B/2)\cos(\pi B/2)\\ &= -\sin^2 B\frac{\pi}{2} + B\frac{\pi^2}{4a}\sin B\pi\end{aligned} \tag{64}$$

Terms in $1/a^2$ could be computed here without much trouble, but they lead to serious complications with the Whittaker's formula, that we want to discuss now.

We first compute the Fourier coefficients of the periodic function J derived from F according to Eq. (14):

$$J = \sum_{n=-\infty}^{+\infty} \theta_n e^{i2nx} \tag{65}$$

We obtain

$$\theta_n = \theta_{-n} = \frac{B^2}{\pi}\int_{-\pi/2}^{\pi/2} \frac{e^{-i2nx}}{(1 + |x|/a)^4}\,dx = \Re e\,\frac{2B^2}{\pi}\int_0^{\pi/2} \frac{e^{-i2nx}}{(1 + x/a)^4}\,dx$$

The result is

$$\theta_0 = \frac{2B^2}{3\pi}\left[1 - \left(1 + \frac{\pi}{2a}\right)^{-3}\right] = B^2\left(1 - \frac{\pi}{a} + \frac{10}{3}\left(\frac{\pi}{2a}\right)^2 \cdots\right),$$
$$\theta_n = \theta_{-n} = \frac{2B^2}{\pi a n^2}\left[1 - (-1)^n + (-1)^n \frac{5\pi}{2a}\right]. \tag{66}$$

Comparing (66) and (63), we note that to the first order

$$S_1 \approx \frac{\pi}{2}\,\theta_0^{1/2}. \tag{67}$$

We also see that θ_n decreases as $1/n^2$ as needed, and that it is of the first order in $1/a$

Whittaker's formula (7) reads

$$\sinh^2\left(\mu\frac{\pi}{2}\right) = -\Delta_1(0)\sin^2\left(\frac{\pi}{2}\,\theta_0^{1/2}\right). \tag{68}$$

We shall prove easily that $\Delta_1(0)$ is practically unity and equation (67) shows that Whittaker's formula (68) checks completely with our solution (64).

Let us now discuss the infinite determinant (8):

$$\Delta_1(0) = |\,B_{mp}\,|, \qquad B_{mm} = 1,$$
$$B_{mp} = \frac{\theta_{m-p}}{\theta_0 - 4m^2}, \qquad (m \neq p). \tag{69}$$

Diagonal elements are all equal to unity while non-diagonal terms are proportional to θ_{m-p}; hence, according to (66), these non-diagonal terms are all very small, of the order $1/a$.

Such a determinant can be computed in the following way: we first take the product of the diagonal elements, that is 1. Next we take all the diagonal elements but two, namely n, n and m, m, which we replace by the non-diagonal terms B_{nm} and B_{mn}, then we take all the diagonals but three (n, n; m, m; p, p) which we replace by B_{nm}, B_{mp}, B_{pm}, etc. Thus, we obtain

$$\Delta = |\,B_{mp}\,| = 1 - \Sigma_{nm}B_{nm}B_{mn} + \Sigma_{n,m,p}B_{nm}B_{mp}B_{pn} - \cdots$$
$$n \neq m \neq p \neq n \cdots\cdots \tag{70}$$

The rule is obvious, and the expansion is ordered with respect to powers of $1/a$, with no term in $1/a$ and terms in $1/a^2$, $1/a^3$ $\cdots$. We decided not to use terms in $1/a^2$ our expansions, hence our determinant is practically unity. Some difficulty may occur when $\theta_0 - 4m^2$ becomes very small (of the order of $1/a$), when the determinant becomes very large. We already noticed in Sec. 1 the inadequacy of Whittaker's formula (68) near the poles of the determinant. There is a compensation, when $\theta_0 = 4m^2$, between the determinant having a double pole and the $\sin^2(\pi/2\,(\theta_0)^{1/2})$ a double zero. Our formula (64) does not exhibit any such trouble.

Otherwise, both methods check completely. It is hoped that the general method developed in this paper will be found useful for practical discussion of many problems reducing to Hill's equation.

—NOTES—

THE SOLUTION OF NATURAL FREQUENCY EQUATIONS BY RELAXATION METHODS*

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The application of Dr. R. V. Southwell's relaxation method[1,2] to the solution of natural frequency equations for systems with a finite number of degrees of freedom is discussed in this article. The first object is to discuss the conditions under which the procedure converges. It is shown that two criteria can be given, such that if the one is satisfied, the procedure converges to the highest mode, and, if the other is satisfied, to the lowest. This is of more than purely theoretical interest: for it is shown that the possibility of finding the highest mode directly can be used to simplify considerably the process of finding further modes.

1. **The criteria of convergence.** The equations to be solved will be written in the form

$$\sum_{s=1}^{n} a_{rs}x_s = p^2 \sum_{s=1}^{n} b_{rs}x_s\,, \qquad (r = 1, 2 \cdots n) \tag{1}$$

where a_{rs} and b_{rs} are constants, $| a_{rs} |$ and $| b_{rs} |$ are symmetrical and the values of p^2 for which a solution is possible must be determined. We write

$$A = A(x) = \sum_{rs} a_{rs}x_rx_s\,, \qquad B = B(x) = \sum_{rs} b_{rs}x_rx_s\,,$$

$$A_r = \frac{\partial A}{\partial x_r}, \qquad B_r = \frac{\partial B}{\partial x_r}, \qquad \lambda = \frac{A}{B}.$$

Both A and B are positive definite forms. In general, λ denotes a variable function of the variables x_s, but the letter λ with subscripts will be used to denote constant values corresponding to specified selections of the variables x.

Suppose that x_k is changed to $x_k + \Delta x_k$. The change in λ is then

$$\Delta\lambda = \frac{A + \Delta A}{B + \Delta B} - \frac{A}{B} = \frac{B\Delta A - A\Delta B}{B(B + \Delta B)} \tag{2}$$

Now,

$$B\Delta A - A\Delta B = B\{A_k\Delta x_k + a_{kk}(\Delta x_k)^2\} - A\{B_k\Delta x_k + b_{kk}(\Delta x_k)^2\} \tag{3}$$

In the relaxation method, a first guess is made to select an initial set of values x, and the corresponding value of λ, say λ_1, is calculated. Then, in the basic application of the method, one of the x_k is adjusted to make $A_k - \lambda_1 B_k$ zero: the x_k corresponding to the largest $A_k - \lambda_1 B_k$ is chosen. (In theory this is the best possible *single* adjustment that may be made; in practice it is rarely used as more *useful* operations can always be performed, according to the skill of the computer, who should always be thinking as

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[1]Southwell, *Relaxation Methods*, Oxford University Press, 1940.

[2]Pellew and Southwell, Proc. Roy. Soc. (A) **175**, 262-290 (1940).

far ahead as possible and therefore using his judgement to modify each adjustment he performs so that it will ultimately have the best possible effect when taken in conjunction with the adjustments which follow). The process is repeated a few times; then the value of λ, corresponding to the values x so obtained, is calculated and used as the new value of λ_1 in the next stage of the process. If the initial values x are correctly chosen, the values of λ_1 found by these successive steps converge to the lowest value of p^2 for which the equations (1) have the solution. We shall now proceed to give a rigorous proof of this last statement, and also to show how the process can, just as easily, be made to converge to the highest mode.

The value of Δx_k, the change in x_k, is determined by the condition that $A_k - \lambda_1 B_k$ vanishes when $x_k + \Delta x_k$ is substituted for x_k. Hence

$$A_k - \lambda_1 B_k + 2(a_{kk} - \lambda_1 b_{kk})\Delta x_k = 0. \tag{4}$$

On substituting this value of Δx_k in (3), we get

$$\begin{aligned} B\Delta A - A\Delta B &= (\Delta x_k)^2\left\{(Ba_{kk} - Ab_{kk}) - \frac{2(BA_k - AB_k)(a_{kk} - \lambda_1 b_{kk})}{A_k - \lambda_1 B_k}\right\} \\ &= -B(\Delta x_k)^2\left\{(a_{kk} - \lambda_1 b_{kk}) + (\lambda - \lambda_1)\left[b_{kk} - \frac{2B_k(a_{kk} - \lambda_1 b_{kk})}{A_k - \lambda_1 B_k}\right]\right\} \end{aligned} \tag{5}$$

after a little algebra, using $A/B = \lambda$.

It is plain from (2) that the sign of $\Delta\lambda$ is the same as that of $B\Delta A - A\Delta B$. Now for the first step in the process, $\lambda = \lambda_1$, and hence from (5) it is obvious that

$$\begin{aligned} \Delta\lambda &< 0 \text{ if } \lambda_1 < a_{kk}/b_{kk}, \\ \Delta\lambda &> 0 \text{ if } \lambda_1 > a_{kk}/b_{kk}. \end{aligned} \tag{6}$$

This leads to the criteria for convergence of the relaxation method:

(i) in order that the process may converge to the lowest mode, the initial values x should be chosen so that the initial value of λ_1 is less than a_{kk}/b_{kk} for all k;

(ii) in order that the process may converge to the highest mode, the initial values x should be chosen so that the initial value of λ_1 is larger than a_{kk}/b_{kk} for all k.

It will be explained shortly how these choices of the initial values x can be made.

If the choice (i) is made, the first step of the relaxation process, with $\lambda_1 = \lambda$, leads to a new value x with a smaller λ. If λ_1 is changed to this smaller λ and the process repeated, it is plain that the value of λ will decrease further. In practice it is not convenient to change λ_1 after each step in the process, and if this change is not made we cannot use (5) to argue directly that a further step will decrease λ still further, for the λ in (5) would be less than λ_1. The sign of the first term in the bracket in (5) will always be positive, but the sign of the term involving $(\lambda - \lambda_1)$ may be positive or negative. However, if λ_1 is near to the lowest proper value the value of $(\lambda - \lambda_1)$ will be small, since from Rayleigh's principle λ must always be greater than the lowest proper value. Even if λ_1 is not near to the lowest proper value, $(\lambda - \lambda_1)$ will be small if the process is not carried on for too many steps with a fixed λ_1. In any case, no matter how many steps of changing an x_k to reduce an $(A_k - \lambda_1 B_k)$ to zero are carried out with a fixed λ_1, λ will not increase above this λ_1; for (5) shows that as soon as λ approaches λ_1 its value will begin to decrease. It is therefore plain that the process will lead to continually

decreasing values of the λ calculated after a series of steps. In practice the relaxation process is continued with a fixed value λ_1, so long as the residuals $(A_k - \lambda_1 B_k)$ as a whole can continue to be decreased. It should also be remarked that for any assumed λ_1, one solution is given by $x_k = 0$ for all k; the computer must avoid approaching this solution by multiplying each x_k by a factor from time to time so that the greatest of the x_k is kept at some constant magnitude.

Similar remarks, with obvious modifications, apply if the choice (ii) is made. λ_1 then increases continually.

Since the λ are bounded above by the highest and below by the lowest characteristic values of the equations, the λ_1 must tend to a limit. This limit must be a proper value of the system, for all the equations (1) are then satisfied. The proper value in question could conceivably be one other than the highest or the lowest, but if this were to occur in actual computation it could do so only as a result of rare good fortune, and by choosing a higher (or lower) value of λ_1 than the proper value thus found we could proceed to find the actual highest or lowest proper values.

2. Practical applications. We shall now show how a suitable set of values x to serve as the first approximation in calculating the highest mode can be found. The method applies equally well, with an obvious change, to the lowest mode, but in most physical problems a good approximation to the lowest mode can be guessed intuitively. Let r and s be the values of m for which a_{mm}/b_{mm} takes on its highest values. The highest value of λ possible with all x_m zero except x_r and x_s is given by the larger root of

$$\begin{vmatrix} a_{rr} - \lambda b_{rr} & a_{rs} - \lambda b_{rs} \\ a_{rs} - \lambda b_{rs} & a_{ss} - \lambda b_{ss} \end{vmatrix} = 0,$$

a quadratic which is easily solved. The corresponding values of x_r and x_s are given by

$$(a_{rr} - \lambda_2 b_{rr})x_r + (a_{rs} - \lambda_2 b_{rs})x_s = 0,$$

where λ_2 denotes the larger root of the quadratic. These values of x_r and x_s, with the other x's zero, should be used as the starting point for the relaxation process.

3. Estimation of frequencies intermediate between the highest and lowest. In the relaxation process described in the references below, it is necessary to find the lowest mode first, and then in finding the other modes to correct the values of the x's found in the successive steps so that they continually satisfy

$$\sum_{rs} a_{rs} x_r^{(1)} x_s = 0, \tag{7}$$

where $x_r^{(1)}$ denotes the value of x_r in the lowest mode. The fact that the highest mode can be found directly saves some time in this process. We shall now show that after finding the highest mode we can alter the problem to a new one so as to make the next highest mode of the old problem the highest mode of the new problem. When this is done, the steps for finding the other modes can be carried out without corrections to satisfy (7).

Let $\lambda^{(i)}$ denote the i-th proper value, $x_k^{(i)}$ the corresponding mode values of x_k, then we have

$$\sum_s a_{rs} x_s^{(i)} = \lambda^{(i)} \sum_s b_{rs} x_s^{(i)}. \tag{8}$$

From this it follows by a well-known argument that if $\lambda^{(i)} \neq \lambda^{(j)}$

$$\sum_{rs} a_{rs} x_r^{(i)} x_s^{(j)} = \sum_{rs} b_{rs} x_r^{(i)} x_s^{(j)} = 0, \tag{9}$$

and even if two proper values are equal we can choose the modes corresponding to them so as to make (9) true. We shall also suppose the modes normalised so that

$$\sum_{rs} a_{rs} x_r^{(i)} x_s^{(i)} = 1. \tag{10}$$

Now consider the matrix $A^{(i)} = | a_{rs}^{(i)} |$, where

$$a_{rs}^{(i)} = \sum_k a_{rk} x_k^{(i)} \sum_e a_{se} x_e^{(i)}. \tag{11}$$

We have

$$\sum_s a_{rs}^{(i)} x_s^{(j)} = \sum_k a_{rk} x_k^{(i)} \sum_{es} a_{se} x_s^{(j)} x_e^{(i)}$$

$$= \begin{cases} 0 & \text{if} \quad i \neq j, \\ \sum_k a_{rk} x_k^{(i)} & \text{if} \quad i = j. \end{cases} \tag{12}$$

Hence, from (8),

$$\sum_s (a_{rs} - a_{rs}^{(i)}) x_s^{(j)} = \begin{cases} 0 & \text{if} \quad j \neq i, \\ \lambda^{(j)} \sum_s b_{rs} x_s^{(j)} & \text{if} \quad j = i. \end{cases}$$

After finding the highest mode and proper value, say $\lambda^{(n)}$, we can form the matrix

$$a_{rs}^1 = a_{rs} - a_{rs}^{(n)}.$$

The highest mode of the original system of equations is annihilated by this matrix, but in the system of equations

$$\sum_s a_{rs}^1 x_s = \lambda \sum_s b_{rs} x_s , \tag{13}$$

all modes save the highest remain proper modes with their former proper values: while the former highest mode now corresponds to the proper value zero. On finding the highest mode of the system (13) by the method described above, we get the second highest mode of the old system. This new mode can then be eliminated in its turn and the mode below it found.

The procedure recommended, therefore, is to find the lowest and highest modes in the normal manner, and then to find the second highest, third highest modes, etc., in order, in the manner described.

It is clear from (12) that the matrix $| a_{rs} - \sum_{i=1}^{n} a_{rs}^{(i)} |$ annihilates all the modes; and since these modes are independent vectors and every vector can be expressed as a linear sum in terms of them, it must annihilate every vector and therefore must be the zero matrix. We have therefore

$$a_{rs} = \sum_{i=1}^{n} a_{rs}^{(i)}.$$

After $(n-1)$ modes have been found, and the reduction of the matrix a_{rs} carried out for each of them in the manner described, we must be left with the matrix

$$a_{rs} - \sum_{j\neq i} a_{rs}^{(j)} = a_{rs}^{(i)},$$

where i is the number of the mode left to be found. In this matrix the terms of any row are proportional to $\sum_s a_{rs}x_s^{(i)}$ so the remaining i-th mode can be found by solving the set of linear equations

$$\sum_s a_{rs}x_s = a_{1r}^{(i)}.$$

The reductions can often be carried out with advantage in terms of the matrix b_{rs}; this is done by normalizing the modes so that instead of satisfying (10) they satisfy

$$\sum_{rs} b_{rs}y_r^{(i)}y_s^{(i)} = 1,$$

and calculating $a_{rs}^{(i)}$ from the formula

$$a_{rs}^{(i)} = \lambda^{(i)} \sum_k b_{rk}y_k^{(i)} \sum_s b_{ss}y_s^{(i)},$$

which is easily deduced from (8). This is particularly convenient in the most usual type of problem in which b_{rs} is a diagonal matrix—in mechanical problems, those in which the kinetic energy can be expressed as a sum of squares. If $| b_{rs} | = | m_r\delta_{rs} |$ the modes must be normalized by

$$\sum_r m_r(y_r^{(i)})^2 = 1$$

and then

$$a_{rs}^{(i)} = \lambda^{(i)}m_rm_sx_r^{(i)}x_s^{(i)}.$$

After reductions corresponding to all but the i-th mode have been made on the matrix a_{rs}, the remaining matrix is $a_{rs}^{(i)}$ from whose rows the i-th mode can be found immediately.

A NORM CRITERION FOR NON-OSCILLATORY DIFFERENTIAL EQUATIONS*

By AUREL WINTNER (*The Johns Hopkins University*)

Let $f(t)$, $x(t)$, $\lambda(t)$, $\cdots$ denote real-valued, continuous functions on an unspecified half-line, $t_0 \leqq t < \infty$. If $\lambda(t)$ is positive on this half-line, put

$$\lambda^* = \lambda^*(t) = \lambda(t) \int_t^\infty (du)/\lambda^2(u), \tag{1}$$

provided that the second factor on the right of (1) is a convergent integral. Under this proviso, a direct substitution of (1) shows that, if $\lambda(t)$ is a solution of the differential equation $D_f(\lambda) = 0$, where

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$$D_f(\lambda) = D_f(\lambda(t)) = \lambda''(t) + f(t)\lambda(t), \qquad (' = d/dt) \tag{2}$$

then $D_f(\lambda^*) = 0$, i.e., that $\lambda^*(t)$ represents another (linearly independent) solution.

Following A. Kneser[1], let the differential equation $D_f(x) = 0$ be called oscillatory or non-oscillatory according as each or none of its solutions $x(t) \not\equiv 0$ has zeros clustering at $t = \infty$. This alternative is complete, since, in view of Sturm's separation theorem, either every or no solution $x(t) \not\equiv 0$ of $D_f(x) = 0$ has an infinity of zeros on the half-line $t \geqq t_0$. The decision of the alternative (for a *given* coefficient function, $f = f(t)$, of D_f) is fundamental in certain questions of stability and related applications[2].

It seems to be of both theoretical and practical interest that the decision can always be based on a criterion similar to the "norm" conditions in the theory of linear functionals and operators (Lebesgue-Toeplitz). It is a criterion the applicability of which does not involve, in principle, the knowledge of a *solution* $x(t) \not\equiv 0$ of $D_f(x) = 0$, since it depends on the consideration of *arbitrary functions*. It can be formulated as follows:

The differential equation $D_f(x) = 0$ is of non-oscillatory type if and only if there exists some positive function, say $\lambda(t)$, corresponding to which the assignments (1), (2) *define two continuous functions the product of which is absolutely integrable, i.e.,*

$$\int^{\infty} \lambda^* \mid D_f(\lambda) \mid dt < \infty. \tag{3}$$

As an illustration of how to apply this criterion, choose the arbitrary function $\lambda(t)$ to be t. Then (1) and (2) reduce to $\lambda^* = 1$ and $D_f(\lambda) = f(t)t$, respectively, and so (3) will be satisfied if $| f(t) | \, t$ has a finite integral over the half-line. It follows that the absolute integrability of $f(t)t$ (which, incidentally, is compatible with $\lim\sup f(t)t = \infty$ *and* $\lim\inf f(t) = -\infty$, where $t \to \infty$) is sufficient in order that the differential equation $D_f(x) = 0$ be of non-oscillatory type.

Actually, this particular sufficient condition is contained in an asymptotic result of Bôcher[3]. But this is not a necessary condition. In fact, other sufficient conditions result if the choice $\lambda = t$ is replaced by other choices of the arbitrary function $\lambda(t)$. Such choices can be made *relative to the coefficient function, f, of D_f*, rather than in a way which, as in $\lambda = t$, is independent of f.

Proof of the sufficiency. This part of the italicized criterion can be deduced from the following fact, which is a corollary of a general theorem[4]: If $p = p(t) \neq 0$ and $q = q(t)$ are continuous functions for large positive t, then the condition

$$\int^{\infty} | q(t) | \left(\int_t^{\infty} | p(u) |^{-1} du \right) dt < \infty \tag{4}$$

is sufficient in order that *some* solution $y = y(t)$ of the differential equation

$$(py')' + qy = 0 \qquad (' = d/dt) \tag{5}$$

[1]A. Kneser, *Untersuchungen über die reellen Nullstellen der Integrale linearer Differentialgleichungen*, Mathematische Annalen **42**, 409-435 (1893), p. 411.

[2]T. v. Kármán and M. A. Biot, *Mathematical methods in engineering*, New York and London, 1940, Chapter VII and the references on p. 322.

[3]M. Bôcher, *On regular singular points of linear differential equations of the second order whose coefficients are not necessarily analytic*, Transactions of the American Mathematical Society **1**, 40-52 (1900), pp. 48-52.

[4]A. Wintner, *Asymptotic integrations of the adiabatic oscillator in its hyperbolic range*, to appear in the Duke Mathematical Journal **15**, (1948).

should tend to a finite limit, as $t \to \infty$, and that this limit, $y(\infty)$, be distinct from 0.

It follows that, if $\lambda = \lambda(t)$ is any positive function possessing a continuous second derivative, then the case

$$p = \lambda^2, \qquad q = \lambda\lambda'' + f\lambda^2 \tag{6}$$

of (5) must have some solution $y = y(t)$ which does not vanish from a certain t onward, if condition (4) is satisfied by the functions (6). But it is clear from the definitions (1), (2) that the case (6) of (4) is identical with (3). Since $\lambda(t)$ is positive, it follows that (3) implies the non-oscillatory character of that differential equation for $x = x(t)$ which results when $y = x/\lambda$ is substituted into the case (6) of (5).

The result of this substitution is seen to be the differential equation

$$(x'\lambda - x\lambda')' + (\lambda'' + f\lambda)x = 0.$$

Since the latter can be contracted into $(x'' + fx)\lambda = 0$, where $\lambda > 0$, it is equivalent to $x'' + fx = 0$ and so, in view of (2), to $D_f(x) = 0$. This completes the proof of the sufficiency of (3).

Proof of the necessity. This part of the criterion is of theoretical interest only, and its verification is straightforward. As a matter of fact, $\lambda(t)$ can now be chosen to be a solution $x(t)$ of $D_f(x) = 0$.

In order to see this, suppose that the differential equation is non-oscillatory. Then there exist a constant t_0 and a solution $x(t)$ of $D_f(x) = 0$ such that $x(t) > 0$ when $t_0 \leqq t < \infty$. Let t^0 be any value exceeding t_0, restrict t to the half-line $t^0 \leqq t < \infty$, and put

$$\lambda(t) = x(t) \int_{t_0}^{t} (du)/x^2(u). \tag{7}$$

Then $\lambda(t)$ is positive, since $x(t)$ is. Furthermore, it is easily verified from (2) and (7) that $D_f(\lambda) = 0$, since $D_f(x) = 0$. Hence, in order to prove that condition (3) is satisfied by the function (7), all that remains to be ascertained is that the function (1) exists in the case (7), i.e., that

$$\int^{\infty} (du)/\lambda^2(u) < \infty \tag{8}$$

holds by virtue of (7). But this can be ascertained by an elementary argument used by Hartman[5].

In fact, it is readily verified from (7) that the Wronskian, $x\lambda' - \lambda x'$, of $x(t)$ and $\lambda(t)$ is the constant 1. Hence, the derivative of x/λ is identical with $-1/\lambda^2$, and so

$$x(t)/\lambda(t) = \text{const.} - \int_{t_0}^{t} (du)/\lambda^2(u).$$

Since $x(t) > 0$ and $\lambda(t) > 0$, it follows that

$$0 < \text{const.} - \int_{t_0}^{t} (du)/\lambda^2(u).$$

This proves (8).

[5] P. Hartman, *On differential equations with non-oscillatory eigenfunctions*, to appear soon.

THE EXTRUSION OF PLASTIC SHEET THROUGH FRICTIONLESS ROLLERS*

By G. F. CARRIER (*Brown University*)

1. Introduction: The Saint Venant-Mises theory of slow plane plastic flow has repeatedly been applied to problems concerning the deformation process which occurs when sheets are formed by passing the material between two fixed cylindrical surfaces with parallel axes. These problems include, of course, the problems of the rolling, extrusion, and drawing of such sheets. The analyses given are of two kinds. In one [1],[1] [6], a simple one-dimensional theory is given; in the other [2], a more laborious scheme is used wherein the flow field is determined numerically by the method of characteristics (now very familiar to engineers because of its use in supersonic aerodynamic theory). It seems desirable to present information which either justifies the use of the simple theory, or modifies the simple theory so that its results become as accurate as those obtained in the numerical process mentioned above. When the thickness t of the formed sheet and the diameter R of the cylindrical forming surfaces are of the same order of magnitude, it appears that the numerical scheme mentioned above is the most efficient procedure. However, when the sheet is thin ($t/R \ll 1$) this procedure becomes very tedious. In this paper, we develop, from the fundamental equation of the Saint Venant-Mises theory, an approximation technique which leads directly to a justification of the one-dimensional theory for the cases where the cylindrical surfaces are frictionless and $t/R \ll 1$. The less idealized case will be treated in a subsequent paper.

2. Formulation of the problem: As is well known [3], the stress analysis in the case of problems of plane plastic strain is based on the yield condition

$$(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 = 4k^2 \tag{1}$$

and the equations of equilibrium

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \tag{2}$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0. \tag{3}$$

The following substitution of variables simplifies the procedure. We set

$$\sigma_x/k = 2\omega + \sin 2\theta \tag{4}$$

$$\sigma_y/k = 2\omega - \sin 2\theta \tag{5}$$

$$\tau_{xy}/k = -\cos 2\theta. \tag{6}$$

We note that this manner of expressing the stress components implies the satisfaction of Eq. (1).

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[1]Numbers in brackets refer to the bibliography at the end of this paper.

Omitting details which can be found in Reference [4], we observe that under fairly general conditions Eqs. (2) and (3) are equivalent to the system

$$y_\xi + x_\eta \cot\theta = 0, \tag{7}$$

$$y_\xi - x_\eta \tan\theta = 0, \tag{8}$$

where

$$\xi = \omega + \theta, \qquad \eta = \omega - \theta.$$

It is evident that this system of equations is linear in the functions x, y which are to be determined as functions of ξ and η. It must be noted that this transformation can be applied only when the region in the x, y plane does not correspond to a line or point (degenerate region) in the ξ, η plane. This will occur in a limited part of our flow region [4].

The boundary conditions are related to the stress conditions at the cylindrical surface and at the inlet and exit sections. In the absence of friction the shear stress τ must vanish at the roll surfaces. On each of these surfaces then

$$\tau(x, y_0) = -\cos\,[2(\theta - \gamma)] = 0,$$

$$\tan\gamma = dy_0/dx,$$

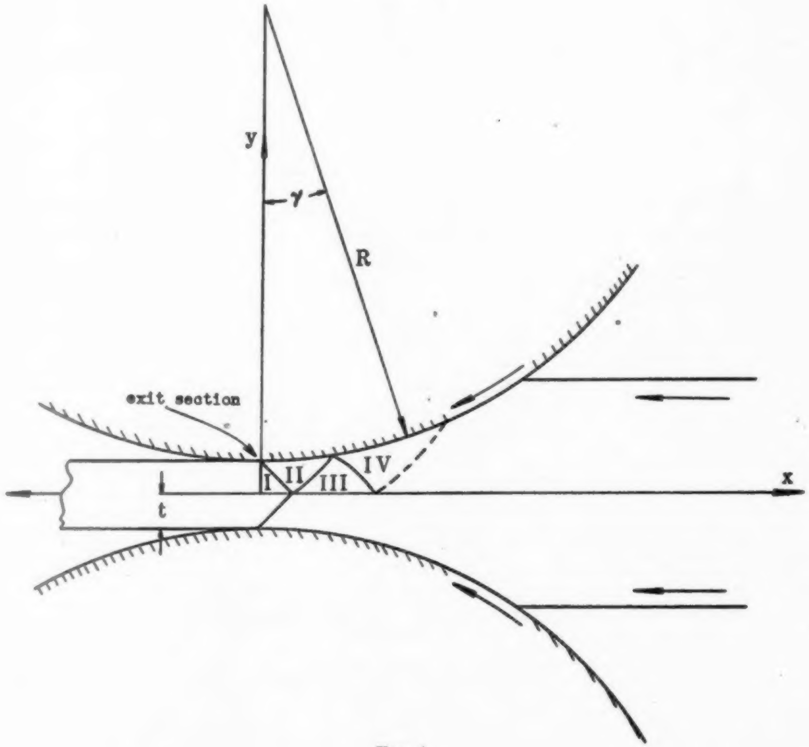

FIG. 1.

where $y = y_0(x)$ is the equation of the roll surface. If we consider that the material enters and leaves as a rigid body (i.e. if we assume the elastic deformations before entrance and after exit to be negligible), we must demand that the material leave the roll (see Fig. 1) with uniform horizontal velocity. Since the velocities must be continuous, a uniform state must then exist in the plastic region adjacent to exit. In fact, it is known (see, for instance, Reference [2]) that region I (Fig. 1) is associated with single values of ξ and η or of ω and θ, namely: $\omega = \omega_{exit} = \omega_e$, $\theta = \pi/4$. The former number is associated with the pull exerted externally on the exit section and the latter value follows from the condition that τ_{xy} must vanish on $y = 0$. It is also known (see for instance Reference [5]) that since region II contains a straight characteristic (its boundary with region I) it is a region in which $\xi = \text{const.} = \pi/4 + \omega_e$, but where η varies.[2] The mapping of the x, y plane onto the ξ, η plane is shown qualitatively in Fig. 2.

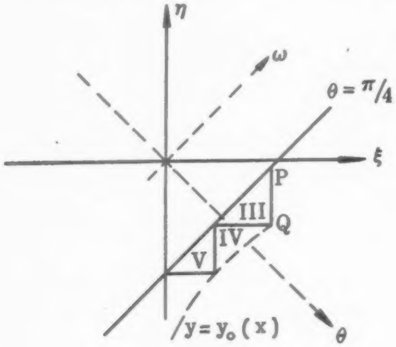

FIG. 2.

In view of the degeneracy of the mapping of regions I and II and in view of the previously stated remark on such mappings, it is evident that any approximate solution of Eqs. (7) and (8) could not apply to regions I and II. We should also note at this point that a boundary value problem of this type cannot have a single analytic solution for the entire region. In fact, the derivatives ξ_x, ξ_y, η_x, η_y will, in general, be discontinuous at the boundaries separating the individual regions. Thus, *it is not possible to write with rigor* either

$$\xi = \sum_0^\infty a_i f_i(x, y), \qquad \eta = \sum_0^\infty b_i f_i(x, y).$$

$$x = \sum_0 \alpha_i \varphi_i(\xi, \eta), \qquad y = \sum_0 \beta_i \omega_i(\xi, \eta),$$

or any other such development.

However, we *can* write

$$x = \sum_0^N \alpha_i \varphi_i(\xi, \eta) = \sum_0^N [g_i(\omega)](\theta - \pi/4)^i \tag{9}$$

$$y = \sum_0^N \beta_i \varphi_i(\xi, \eta) = \sum_0^N [f_i(\omega)](\theta - \pi/4)^i \tag{10}$$

[2]This remark would be slightly modified for the case where non-vanishing wall friction exists, but it applies rigorously to our present problem.

where *by proper choice of* N, *the* f_i, *and the* g_i *we can make these functions approach as closely as we wish*[3] *the functions* $x(\omega, \theta)$ *and* $y(\omega, \theta)$. We observe that the actually existing discontinuous derivatives will not appear in these functions, but we also note that no differentiation of these functions is required in order to give the complete state of stress.

The equation of the upper roll boundary can be written as

$$y_0(x)/R = \epsilon + 1 - (1 - x^2/R^2)^{1/2} = \epsilon + 1 - \cos \gamma.$$

On the roll surface (i.e. at $y = y_0$)

$$\tau = -\cos 2(\theta - \gamma) \equiv 0,$$

i.e.

$$\theta = \pi/4 + \gamma, \tag{11}$$

and on $y = 0$, τ_{xy} vanishes; that is, on $y = 0$, $\theta = \pi/4$. We now let the quantities x, y in Eqs. (9) and (10) be the dimensionless coordinates[4] x/R, y/R, so that on the roll boundary we have [from Eq. (11)]

$$\theta = \pi/4 + \gamma,$$
$$y = \epsilon + 1 - \cos \gamma,$$
$$x = \sin \gamma,$$

that is,

$$\sin \gamma = \sum_0^N g_i(\omega)(\theta - \pi/4)^i = \sum_0^N g_i(\omega)\gamma^i, \tag{12}$$

$$\epsilon + 1 - \cos \gamma = \sum_{i=1}^N f_i(\omega)\gamma^i. \tag{13}$$

We now note that Eqs. (7) and (8) can be written

$$(1 + \tan \gamma)y_\xi + (1 - \tan \gamma)x_\xi = 0,$$
$$(1 - \tan \gamma)y_\eta - (1 + \tan \gamma)x_\eta = 0,$$

or

$$(1 + \gamma + \gamma^3/3 + \cdots)y_\xi + (1 - \gamma + \cdots)x_\xi = 0, \tag{14}$$

$$(1 - \gamma - \cdots)y_\eta - (1 + \gamma + \cdots)x_\eta = 0, \tag{15}$$

and when we use Eqs. (9) and (10), we obtain [noting that $y(\omega)$ is odd in $\theta - \pi/4$]

$$f_1 = -g_0', \tag{16}$$

$$g_2 = g_0' + g_0''/2, \tag{17}$$

$$f_3 = \frac{1}{3}(2g_0' + g_0'' - g_0'''/2), \tag{18}$$

[3]In the integral mean square sense.
[4]R is the radius of the cylinder.

$$g_4 = \frac{1}{4}\left(\frac{2g_0'}{3} - \frac{4g_0''}{3} + \frac{g_0'''}{3} + \frac{g_0^{\text{iv}}}{3}\right), \tag{19}$$

. . . .

When these are substituted in Eqs. (14), (15) we obtain two differential equations in the functions $\gamma(\omega)$, $g_0(\omega)$. These equations can be written in the form

$$\begin{aligned}\frac{\gamma\gamma'}{\gamma^2/2+\epsilon} + 1 = \frac{1}{(\gamma^2/2+\epsilon)}\Big\{&\left(\frac{4}{3}g_0'' + \frac{2}{3}g_0' - \frac{1}{3}g_0'''\right)\gamma^3 \\ &+ \left(\frac{2}{9}g_0' - \frac{1}{45}g_0'' - 23g_0''' + \frac{1}{6}g_0^{\text{iv}} + \frac{1}{60}g_0^{\text{v}}\right)\gamma^5 \\ &+ 2\left(g_0 + \frac{1}{2}g_0''\right)\gamma^2\gamma' + \left(\frac{2}{3}g_0' - \frac{4}{3}g_0'' + \frac{1}{3}g_0''' + \frac{1}{3}g_0^{\text{iv}}\right)\gamma^4\gamma' + \cdots\Big\},\end{aligned} \tag{20}$$

$$\begin{aligned}\frac{g_0g_0'}{g_0^2/\epsilon+1} + 1 = \frac{1}{2(g_0^2/2+\epsilon)}\Big\{&-(2g_0' + g_0)\left(g_0' + \frac{1}{2}g_0''\right)\gamma^2 \\ &- \left(\frac{1}{3}g_0' + \frac{1}{6}g_0'' + \frac{1}{3}g_0'''\right)\gamma^3 - \frac{1}{4}(2g_0' + g_0)\Big(\frac{2}{3}g_0' - \frac{4}{3}g_0'' \\ &+ \frac{1}{3}g_0''' + \frac{1}{3}g_0^{\text{iv}}\Big)\gamma^4 + \left(\frac{5}{18}g_0' - \frac{1}{45}g_0'' - \frac{13}{180}g_0''' + \frac{1}{12}g_0^{\text{iv}} - \frac{2}{15}g_0^{\text{v}}\right)\gamma^5 + \cdots\Big\}.\end{aligned} \tag{21}$$

We note that these equations are of an order which depends on the value of N [see Eq. (12)]. Thus, many boundary conditions are indicated. Recalling, however, the given information concerning regions I, II, we see that at $\theta = \pi/4$ [i.e. $y = 0$], $g_0(\omega_e) = \epsilon$. Also at $\gamma = 2\epsilon$ (the upper corner of region II), we have $\omega + \theta = \omega_e + \pi/4$, $\theta = \pi/4 + 2\epsilon$ hence $\omega = \omega_e - 2\epsilon$. If we now define $s = \omega_e - \omega$, the differential equations (20), (21) become

$$\begin{aligned}\frac{\gamma\gamma'}{\epsilon+\gamma^2/2} = 1 - \frac{1}{(\gamma^2/2+\epsilon)}\Big\{&\left(\frac{4}{3}g_0'' + \frac{2}{3}g_0' - \frac{1}{3}g_0'''\right)\gamma^3 \\ &+ \left(\frac{2}{9}g_0' - \frac{1}{45}g_0'' - 23g_0''' + \frac{1}{6}g_0^{\text{iv}} + \frac{1}{60}g_0^{\text{v}}\right)\gamma^5 \\ &+ 2\left(g_0 + \frac{1}{2}g_0''\right)\gamma^2\gamma' + \left(\frac{2}{3}g_0' - \frac{4}{3}g_0'' + \frac{1}{3}g_0''' + \frac{1}{3}g_0^{\text{iv}}\right)\gamma^4\gamma' + \cdots\Big\},\end{aligned} \tag{22}$$

$$\begin{aligned}\frac{g_0g_0'}{g_0^2/\epsilon+1} = 1 - \frac{1}{2(g_0^2/2+\epsilon)}\Big\{&-(2g_0' + g_0)\left(g_0' + \frac{g_0''}{2}\right)\gamma^2 \\ &- \left(\frac{1}{3}g_0' + \frac{1}{6}g_0'' + \frac{1}{3}g_0'''\right)\gamma^3 - \frac{1}{4}(2g_0' + g_0)\Big(\frac{2}{3}g_0' - \frac{4}{3}g_0'' \\ &+ \frac{1}{3}g_0''' + \frac{1}{3}g_0^{\text{iv}}\Big)\gamma^4 + \left(\frac{5}{18}g_0' - \frac{1}{45}g_0'' - \frac{13}{180}g_0''' + \frac{1}{12}g_0^{\text{iv}} - \frac{2}{15}g_0^{\text{v}}\right)\gamma^5 + \cdots\Big\},\end{aligned} \tag{23}$$

where the primes now indicate differentiation with respect to s.

Rigorously, we have more boundary conditions along the line ξ = const. which represents region II of the physical plane. If these were applied, the problem would be over-determined and, in fact, we could at best obtain a solution valid for region III only. If we relax these conditions and define $\varphi = \ln(1 + \gamma^2/2\epsilon)$, $\psi = \ln(1 + g_0^2/2\epsilon)$, Eqs. (22) and (23) can be put into the form

$$
\begin{aligned}
\varphi = s - \int_0^s \frac{1}{(\gamma^2/2 + \epsilon)} \Big\{ & \Big(\frac{4}{3}g_0'' + \frac{2}{3}g_0' - \frac{1}{3}g_0'''\Big)\gamma^3 \\
& + \Big(\frac{2}{9}g_0' - \frac{1}{45}g_0'' - 23g_0''' + \frac{1}{6}g_0^{\mathrm{iv}} + \frac{1}{60}g_0^{\mathrm{v}}\Big)\gamma^5 \\
& + 2\Big(g_0 + \frac{1}{2}g_0''\Big)\gamma^2\gamma' + \Big(\frac{2}{3}g_0' - \frac{4}{3}g_0'' + \frac{1}{3}g_0''' + \frac{1}{3}g_0^{\mathrm{iv}}\Big)\gamma^4\gamma' + \cdots \Big\}\, ds,
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
\psi = s + \int_0^s \frac{1}{2(g_0^2/2 + \epsilon)} \Big\{ & (2g_0' + g_0)\Big(g_0' + \frac{g_0''}{2}\Big)\gamma^2 \\
& + \Big(\frac{1}{3}g_0' + \frac{1}{6}g_0'' + \frac{1}{3}g_0'''\Big)\gamma^3 + \frac{1}{4}(2g_0' + g_0)\Big(\frac{2}{3}g_0' - \frac{4}{3}g_0'' \\
& + \frac{1}{3}g_0''' + \frac{1}{3}g_0^{\mathrm{iv}}\Big)\gamma^4 - \Big(\frac{5}{18}g_0' - \frac{1}{45}g_0'' - \frac{13}{180}g_0''' + \frac{1}{12}g_0' - \frac{2}{15}g_0\Big)\gamma^5 + \cdots \Big\}\, ds
\end{aligned} \tag{25}
$$

The equations are of the conventional non-linear Voltera type and may be treated by successive approximations. Without going into details, it may be stated that the first approximation and the subsequent ones are in excellent agreement for $s > 2\epsilon$, whereas in the neighborhood of $s = 0$, the behavior is fairly erratic. However, $\gamma(s)$ is of interest only for $s \geq 2\epsilon$ and $g_0(s)$ only for $s > \epsilon$. Hence, the solution

$$\varphi(s) \simeq s, \qquad \psi(s) \simeq s \tag{26}$$

provides an excellent estimate of the state of stress existing in the flow field. We can write in fact, [this is equivalent to Eq. (26)]

$$\omega_e - \omega = \ln(1 + \gamma^2/2\epsilon) \tag{27}$$

along the boundary, and

$$\omega_e - \omega = \ln(1 + g_0^2/2\epsilon) = \ln(1 + x^2/2\epsilon) \tag{28}$$

along $y = 0$. These formulae are good for $s > 2\epsilon$ and are essentially Sachs' formulae [1].

The corresponding results for the case where the wall friction does not vanish require certain numerical work and will be presented in a later paper. Detailed interpretative remarks concerning the foregoing result do not seem to be in order here since they would necessarily coincide with the findings in [1] regarding this specific problem.

References

1. G. Sachs and L. J. Klinger, *The flow of metals through tools of circular contour*, J. Appl. Mech. **14**, No. 2 (1947).

2.* K. H. Shevchenko, *The plastic state of stress and the flow of metals in cold rolling and drawing,* Izvestia Ak. Nauk SSSR, OTN **1946**, No. 3, pp. 329-354.

3. See, for example, A. Nadai, *Plasticity*, McGraw Hill Book Co., Inc., New York, 1931.

4.* S. Khristianovich, *The plane problem of the mathematical theory of plasticity in the case where the external forces are given along a closed contour*, Mat. Sbornik 1, 511-534 (1936).

5. R. Courant, *Supersonic flow and shock waves*, Applied Mathematics Panel, Report 38.2R, p. 39.

6.* A. Iu. Ishlinskii, *Rolling and drawing at high speeds*, Prikl. Mat. Mekh. **7**, 226-230 (1943).

ADDITIONAL CORRECTIONS TO OUR PAPER

THE CYLINDRICAL ANTENNA: CURRENT AND IMPEDANCE

QUARTERLY OF APPLIED MATHEMATICS **3**, 302-335 (1946) and **4**, 199-200 (1946)

BY RONOLD KING AND DAVID MIDDLETON (*Harvard University*)

page 305, Change the number of Eq. (13a) to (13); delete "so that" following Eq. (13a); delete Eq. (13b).

page 306, Change the number of Eq. (14a) to (14); delete "where" following Eq. (14a); delete Eq. (14b).
Eq. (16)—add superscript -1 on R_{1h} in the integrand.

page 317, Fig. 10—The value $|\psi_2(h - \lambda/4)|$ should be at 16.6 instead of 17.4 with appropriate changes in the several curves.

page 321, Figs. 12 and 13—All the curves are somewhat in error for $\beta h < \pi/2$. The correct values are obtained from (74), using the corrected values for ψ obtained from Fig. 11a on page 200 of volume 4.

page 327, Table II—First line: Insert $-$ between $\pi/2$ and βh_{res}.
Second line: Replace 800 by 820.
Fourth line: Replace 67 by 73.

page 328, Eqs. (14a), (14b)—Insert $-$ after $=$.

page 330, Eq. (23)—Change sign of lower limits on all three integrals by inserting $-$ sign. This change is in addition to corrections on page 200 of volume 4.
Eq. (27)—Change first $-$ sign to $+$; change last $+$ sign to $-$.

page 335, Eq. (45)—Change all upper limits in four integrals to u_2.
Change all lower limits in four integrals to $-u_1$.
Eq. (46)—Last integral only: Change upper limit to u_2, lower limit to $-u_1$.
Eq. (47)—Delete superscript bars in second integral of the first member of the equation. Change $-$ to $+$ before this second integral.

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*These articles have been translated for the David W. Taylor Model Basin, United States Navy, by the Applied Mathematics Division, Brown University, Providence, R. I.

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